LETTER TO THE EDITOR

Search for the spin glass transition using series methods

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Abstract. High-temperature series expansions for the nearest-neighbour Ising model of a spin glass are derived. The concentration dependence of the critical temperature is analysed by Padé methods, and no indication of a spin glass transition is observed. The analysis does not rule out cusplike behaviour.

The Ising hamiltonian

\[ \mathcal{H} = - \sum_{NN} J_{ij} s_i s_j - H \sum_i s_i, \quad (1) \]

has, over the past couple of years, been the subject of considerable attention as a simple model of a spin glass (Fischer 1977). The constant nearest-neighbour interactions \( J_{ij} \) of the regular Ising model are replaced by interactions of random strength and sign, subject only to an overall distribution \( P(J) \), in an attempt to simulate the effective interactions between magnetic impurities in a dilute alloy. The properties of this system, and its Heisenberg extension, have been studied by various mean field (Edwards and Anderson 1975, Sherrington and Kirkpatrick 1975, Klein 1976) and renormalisation group (Harris et al 1976, Jayaprakash et al 1977) techniques, and by the Monte Carlo method (Binder and Schröder 1976, Binder and Stauffer 1976).

In the case of a Gaussian distribution of interaction strengths \( P(J) \), with width \( \Delta J \) and centred about a point \( J_o \geq 0 \), the mean field analysis indicates the existence of a 'spin glass' ordered phase for \( J_o/\Delta J \) below a certain value, and beyond this a ferromagnetic ground state. The spin glass state is characterised by zero magnetisation, \( m = \langle s \rangle = 0 \), but the more general order parameter \( q = \langle s^2 \rangle \) is nonzero; here \( \langle \ldots \rangle \) denotes the spin configurational average for a particular arrangement of interactions \( \{ J_{ij} \} \), and the bar denotes the average over all possible sets \( \{ J_{ij} \} \) distributed according to \( P(J) \). The onset of the spin glass phase is marked by cusps in both the zero-field susceptibility and specific heat. The existence of the susceptibility cusp in two and three dimensions is confirmed by the Monte Carlo studies, but the specific heat appears to be smoothly rounded, a result more in accord with experiment (Fischer 1977). It should be noted that the underlying models used in the Monte Carlo and mean-field studies are not identical: in the latter the nearest neighbour sum in (1) is replaced by a sum over all spin pairs, and the interactions \( J_{ij} \) (which must be scaled by the lattice size) are taken to be independent of the distances between the spins.

An alternative to the Gaussian distribution is one in which the interactions can take
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only two distinct values $+J$ and $-J$, namely

$$P(J') = p \delta(J' - J) + (1 - p) \delta(J' + J) \quad 0 \leq p \leq 1. \quad (2)$$

For $p$ near unity one can reasonably expect the system to order ferromagnetically at low temperature, and antiferromagnetically for $p$ near zero. Spatial renormalisation methods (Jayaprakash et al. 1977) confirm this view and, furthermore, predict the existence of a spin glass phase at intermediate $p$ in $d = 2$, 3 and 4 dimensions, but the validity of this approach has been questioned (Kirkpatrick 1977). Series analysis of a generalised spin glass susceptibility (Fisch and Harris 1977) provides evidence of a transition for $4 < d < 6$, but the analysis breaks down below $d = 4$. A modification of the model in which each $J_{ij}$ is expressed as a product of two random variables $\epsilon_i$ and $\epsilon_j (= \pm 1)$, each associated with a single lattice site only, has also been proposed (Mattis 1976, Aharony and Imry 1976). This model has Ising-like properties for all $p$ (defined via $\epsilon_i = 2p - 1$), except at $p = 0.5$ where the singularity in the susceptibility is reduced to a cusp. However the transformation $J_{ij} \rightarrow J\epsilon_i\epsilon_j$ is strictly correct only for a lattice with no closed loops (the Bethe lattice), and there is no way of using the results obtained from this approximation to predict the properties of the original system.

In this note we report on a study of the Ising system defined by (1) and (2) using series methods. The quantities most amenable to study and, because of their strongly singular behaviour in the $p = 1$ limit, most likely to reveal how the behaviour depends on $p$, are the zero-field susceptibility $\chi$ and its higher order field derivatives $\chi^{(2)}, \ldots$. The free energy is given by $F = -T \ln \text{tr} \exp (-\mathcal{H}/T)$, and $\chi = \partial^2 F / \partial H^2$, $\chi^{(2)} = \partial^2 \chi / \partial H^2$, the quantities to be evaluated at $H = 0$.

The techniques used to generate series for random bond Ising systems have been described in detail elsewhere (Rapaport 1972a, b) for a model in which the interaction strength was either $J$ or zero. The only modification required for the $\pm J$ case is a slight change in the algorithm used to determine the weights of the contributing graphs. The final series are expressed in the form:

$$\chi = 1 + \sum_{n=1}^{N} \sum_{m=0}^{n} A_{mn}(2p - 1)^mw^n, \quad w = \tanh J/T$$

and likewise for $\chi^{(2)}$, except that the leading term is $-2$. (Factors of $1/T$ and $1/T^3$ multiplying the two series are omitted for convenience; they do not affect the subsequent analysis.) The series were generated for the loose-packed body-centred cubic (bcc) lattice with $N = 12$ for $\chi$, and $N = 11$ for $\chi^{(2)}$. For $p = 1$ the $\chi$ series checks against the regular Ising result (Domb 1974), and $\chi^{(2)}$ agrees with the corresponding high-temperature expansion which can be derived from low-temperature series data (Domb 1974). The coefficients $A_{mn}$ for the two series are listed in tables 1 and 2.

The model under study is a ‘quenched’ system, in the sense that the interactions $J_{ij}$ are frozen in place in a random manner. There exists another model, having the same hamiltonian, but with the $J_{ij}$ permitted to rearrange themselves on the lattice bonds so as to achieve complete thermodynamic equilibrium with the spins—the ‘annealed’ system. The latter can be expressed in terms of a regular (i.e., $p = 1$) Ising model with temperature dependent interactions (Kasai and Syozi 1973), and is found to have a second-order transition with renormalised exponents for $0 \leq p \leq 1 - p'$ and $p' \leq p \leq 1$. The critical concentration $p'_s$ is known exactly for $d = 2$ and numerically for $d = 3$, and there is no evidence for a spin glass transition at intermediate $p$. The difference between the quenched and annealed systems is that in the latter the free energy is given by
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\[ F' = -T \ln \left( \text{tr} \exp \left(-\mathcal{H}/T\right) \right) \] and, though there is no reason to expect any connection between \( F \) and \( F' \), it will prove interesting to compare the observed properties of the two systems.

The series were analysed by means of Padé approximants (Hunter and Baker 1973). If the assumed asymptotic behaviour of \( \chi \) (or \( \chi^{(2)} \)) is

\[ \chi(T, p) \sim A(p) \left( w_c(p) - w \right)^{-\gamma(p)}, \quad w_c(p) = \tanh J/T_c(p) \]

then \( d/dw \ln \chi \) should have a simple pole at \( w_c(p) \) with residue \( \gamma(p) \). The \( w_c(p) \) estimates obtained from such an analysis of the \( \chi \) and \( \chi^{(2)} \) series are given in table 3. Also shown are the results of a similar analysis of 12 terms of the \( \chi \) expansion for the annealed system, together with the true annealed \( w_c(p) \) values. The absence of an entry in the table means that the Padé approximants concerned failed to produce a pole on the positive real axis. Note that the results are symmetric about \( p = 0.5 \).

What is immediately apparent is that the critical temperature estimates based on \( \chi \) and \( \chi^{(2)} \) of the quenched system and those of the annealed system are practically identical and that, at least in the annealed case, the Padé estimates are accurate. The complete annealed \( T_c(p) \) curve is shown in figure 1; the points corresponding to \( T_c(p) \) for the quenched system deviate from this curve by less than the thickness of the line. The ferromagnetic branch of the annealed curve reaches \( T = 0 \) at \( p_c = 0.636 \), and there is no hint that \( T_c(p) \) for the quenched system does not continue to decrease smoothly and reach zero at a

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**Table 1.** The coefficients \( A_{nm} \) of the \( \chi \) expansion. For each \( n \) the values are given as a sequence of pairs \( m: A_{nm} \).

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value $p = p_s$ close to $p_c$. The similarity between quenched and annealed systems was previously noted for the dilute bond ferromagnet (Rapaport 1972a, b) where it was pointed out that there exists a small but definite difference in the $T_c(p)$ behaviour; on this basis the values of $p_s$ (presuming that it exists) and $p_c$ need not be identical.

The results point to ferro- and antiferromagnetic behaviour with a singularity at $T_c(p)$ for $p > p_s$ and $p < 1 - p_s$, respectively. The near equality of the quenched and annealed predictions suggests that the quenched $T_c(p)$ approaches zero, with no sign of the spin-glas transition predicted by spatial renormalisation methods. There still remains the possibility that, at intermediate $p$, a transition with a cusp in $\chi$ occurs; here there is no need for a singularity in $\chi$ because the cusp can arise merely from the intersection of the high- and low-temperature branches of the function. Behaviour of this kind is unde-
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**Figure 1.** Variation of the critical temperature with $p$. The full curves are for the annealed version of the model—the quenched results also lie on these curves. The ferro-, antiferro- and paramagnetic phases are indicated.

Detectable by currently available series analysis techniques. In particular, for $p = 0.5$ the high-temperature susceptibility has the simple form $\chi = 1/T$ (Domb 1976); if a spin glass transition really does occur in this system it cannot be marked by a singularity in the high-temperature branch of $\chi$. A calculation which predicts that at $p = 0.5$ $\chi^{(2)}$ is singular at $w_c(1)^{1/2}$ (Domb 1976) does not represent the situation correctly; the predicted values of the series coefficients do not agree with those of table 2 because certain important classes of graphs were omitted from the analysis (C Domb, private communication).

We have not presented detailed results for the behaviour of the exponents $\gamma(p)$. The Padé results for both $\chi$ and $\chi^{(2)}$ show steadily increasing exponent values as $p$ decreases from unity. Similar behaviour was encountered in the dilute magnet (Rapaport 1972a) and, at least in the case of the annealed version, is attributable to confluent singularities. The accuracy of the $w_c(p)$ estimates does not appear to be affected by the more complex singularity structure however.

In summary, the series results suggest that the singularities in $\chi$ and $\chi^{(2)}$ approach $T = 0$ as $p \to p_s + 0$ and $p \to (1 - p_s) - 0$, and no evidence of a crossover to any other kind of phase transition is observed. The possibility of a spin-glass transition involving a cusp at finite $T$ is not excluded by the analysis. We are currently investigating this model using Monte Carlo techniques in the hope of learning what really happens, and whether the behaviour resulting from distribution (2) differs substantially from that of the Gaussian distribution.

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