Representation of Finite Games as Network Congestion Games

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Abstract. Weighted network congestion games are a natural model for interactions involving finitely many non-identical users of network resources, such as road segments or communication links. However, in spite of their special form, these games are not fundamentally special: *every* finite game can be represented as a weighted network congestion game. The same is true for the class of (unweighted) network congestion games with player-specific costs, in which the players differ in their cost functions rather than their weights. The intersection of the two classes consists of the unweighted network congestion games. These games *are* special: a finite game can be represented in this form if and only if it is an exact potential game.

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1 Introduction

In a (finite) congestion game, finitely many players share a finite set E of resources but may differ in which resources they are allowed to use. Specifically, each player's set of strategies is a particular collection of nonempty subsets of E. The player's payoff from using a strategy is the negative of the total cost of using the resources included in the strategy. The cost of a resource depends only on its identity and on the number of users. The cost does not necessarily increase with congestion, and it may be negative, in which case using the resource contributes positively to the payoff. Rosenthal (1973) showed that every congestion game admits an exact potential, which is a function P on strategy profiles that exactly reflects the players' incentives to unilaterally change their strategies. Whenever a single player moves to a different strategy, his gain or loss is equal to the corresponding change in P. Monderer and Shapley (1996) proved the converse: essentially, *only* congestion games are exact potential games. More precisely, every finite game that admits an exact potential can be represented as (in other words, it is isomorphic to) a congestion game. One of this paper's findings strengths Monderer and Shapley's result by showing that an exact potential game can always be represented as a particular kind of congestion game, namely, an unweighted network congestion game.

A restriction or expansion of the meaning of 'congestion game' potentially has the same effect on the class of representable games. Examples of restriction are: congestion games with nondecreasing cost functions, in which increasing congestion never makes users better off; singleton congestion games, in which each strategy includes a single resource; and

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network congestion games, in which resources are represented by edges in a graph and strategies correspond to routes, which are paths in the graph that connect the player's designated origin and destination vertices. (The second restriction is a special case of the third. It corresponds to a parallel network, which is one with only two vertices.) Examples of extensions are: congestion games with player-specific costs (or payoffs, Milchtaich 1996), where players are differently affected by congestion; and weighted congestion games, where their contributions to it (the players' "congestion impacts") differ.

This paper shows that both weighted network congestion games and (unweighted) network congestion games with player-specific costs are actually capable of representing all finite games. Both representations use only nondecreasing (but not necessarily positive) cost functions and two-terminal, or single-commodity, networks, in which all players' routes start and terminate at the same origin and destination vertices. Thus, although the definitions of the two kinds of network congestion games involve quite specific structures, the games themselves are not in any way special. For unweighted network congestion games, which are simultaneously weighted network congestion games and network congestion games with player-specific costs, this is not so. As indicated, these games are special in that they represent (all) exact potential games.

The potential significance of these representation results lies in the properties of the represented game that can be inferred from *partial* information about the representation, in particular, information about the network used. An example of such a property is existence of pure-strategy Nash equilibrium. As the proof of Theorem 1 below shows, every 2×2 game can be represented both as a weighted network congestion game and as a network congestion games with player-specific costs on the network depicted in Fig. 1b. By contrast, a representation of either kind on the network in Fig. 1c may exist only if the game has at least one pure-strategy equilibrium (Milchtaich 2012). Thus, the fact that the game in Fig. 1a, for example, has such an equilibrium can be inferred from the fact it has such a representation. No additional information about the game in Fig. 1a is needed – not even the number of players.

2 Preliminaries

2.1 Game Theory

A finite (noncooperative) game Γ has a finite number n of players, numbered from 1 to n. Each player i has a finite strategy set S_i^Γ and a payoff function h_i^Γ that specifies i's payoff for each strategy profile $(s_1, s_2, \dots, s_n)$. For two finite games Γ and Γ' with the same number n of players, a *homomorphism* from Γ to Γ' is

(i) a renumbering of the players in Γ' and

(ii) a function $\phi_i: S_i^\Gamma \to S_i^{\Gamma'}$ from the strategy set of each player i in Γ to that of player i (according to the new numbering) in Γ'

such that for every strategy profile $(s_1, s_2, \dots, s_n)$ in Γ

$$h_i^\Gamma(s_1, s_2, \dots, s_n) = h_i^{\Gamma'}(\phi_1(s_1), \phi_2(s_2), \dots, \phi_n(s_n)), \qquad i = 1,2, \dots, n.$$

The functions ϕ_i are not necessarily bijections.[1] If they are, the homomorphism is an *isomorphism*, and if in addition Γ and Γ' are the same game, it is an *automorphism* (Nash 1951).[2] Two games with the same number of players are *isomorphic* (Monderer and Shapley 1996) if there is an isomorphism from one to the other. Such games are essentially just alternative representations of a single game. Two games Γ and Γ' with identical sets of players and respective strategy sets are *similar* if, for every strategy profile, the change in the payoff of any player who unilaterally switches to any alternative strategy is the same in both games. Put differently, similarity means that, for each player i, the difference $h_i^{\Gamma} - h_i^{\Gamma'}$ between i's payoffs in the two games can be expressed as a function of the other players' strategies. A game Γ is an *exact potential game* (Monderer and Shapley 1996) if it is similar to some game Γ' in which all the players have the same payoff function; that function P is said to be an *exact potential* for Γ.

2.2 Graph Theory

An *undirected multigraph* consists of a finite set of vertices and a finite set of edges. Each edge e joins two distinct vertices, u and v, which are referred to as the *end vertices* of e. Thus, loops are not allowed but more than one edge can join two vertices. An edge e and a vertex v are *incident* with each other if v is an end vertex of e. A (simple) *path* of length m is an alternating sequence of vertices and edges $v_0 e_1 v_1 \cdots v_{m-1} e_m v_m$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it and all the vertices (and necessarily all the edges) are distinct. Every path traverses each of its edges e in a particular *direction*: from the end vertex that immediately precedes e in the path to the vertex that immediately follows it.

A *two-terminal network*, or simply *network*, G is an undirected multigraph together with a distinguished ordered pair of (distinct) *terminal* vertices, the *origin* o and the *destination* d, such that each vertex and each edge belongs to at least one path that begins with o and ends with d. Such a path is called a *route* in G.

A network G may be connected with another network G', which does not share any of its vertices and edges, *in parallel* or *in series*. The sets of vertices and edges in the resulting network are the unions of the corresponding sets in G and G', except that, for a connection in parallel, the two origin vertices are identified and the two destination vertices are identified, and for a connection in series, the destination in G and the origin in G' are identified and become a non-terminal vertex. The connection of an arbitrary number of networks in parallel or in series is defined recursively.

[1] An example is identification of equivalent strategies, which lumps them into a single strategy in the reduced normal form of the game.

[2] The definitions of the various morphisms extend in a natural manner to games that are not necessarily finite, to incomplete information games and to games with random player sets (see Milchtaich 2004). These definitions and their extensions comply with the axioms of category theory.

2.3 Network Congestion Games

A *weighted network congestion game* on a (two-terminal[3]) network G is a finite, n-player game that is defined as follows. First, each edge in G is assigned a nondecreasing *cost function*[4] $c_e(0,\infty) \to (-\infty,\infty)$, an allowable direction, which must be that in which some route in G traverses the edge, and a (possibly, empty) set of allowable users. An edge is *public* or *private* if it allowable to all players or to one player only, respectively. It is required that each player i has at least one *allowable route*, that is, a route in G that includes only edges that i is allowed to use and traverses them in the allowable direction. The collection of all such routes is the player's strategy set S_i. Second, a *weight* $w_i > 0$ is specified for each player i, which represents the player's congestion impact.[5] The total weight f_e of the players whose chosen route includes an edge e is the *flow* (or load) on e. The *cost* of e for each of its users is $c_e(f_e)$. A player's payoff in the game is the negative of the total cost of the edges in his route.

A weighted network congestion game is referred to as an *unweighted* network congestion game if the players' weights are all identical and equal to 1. The equality of the weights entails, in particular, that the cost of an edge is not affected by the identities of its users but only by their number. A generalization that allows for a dependence of the cost for a user on his own identity is (unweighted) *network congestion game with player-specific costs*. In such a game, each edge e is associated with a (player-specific) nondecreasing cost function $c_{ie} : (0,\infty) \to (-\infty,\infty)$ for each player i, and its cost for that player is $c_{ie}(f_e)$, where (the flow) f_e is the total number of players using e.

3 Representation Results

Every unweighted network congestion game is in particular a congestion game in the sense of belonging to the class of games presented by Rosenthal (1973) and studied by Monderer and Shapley (1996) (see Section 1). In fact, as the following theorem shows, the class of unweighted network congestion games essentially *coincides* with that of all congestion games. By contrast, weighted network congestion games and network congestion games with player-specific costs are generally not congestion games in the above sense. The theorem shows that both classes essentially coincide with that of *all* finite games.

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[3] The assumption of a single origin–destination pair may be viewed as a normalization. Any weighted network congestion game on a *multi-commodity network*, which has multiple origin–destination pairs, may also be viewed as a game with a single such pair. In that game, each terminal vertex is incident with a single allowable edge (see below) for each player, which joins it with the player's corresponding terminal vertex in the original game. The interesting special case in which these are the *only* edges that not everyone is allowed to use is not considered in this paper.

[4] A negative cost is interpreted as (net) *gain*. It would also be possible to use the opposite sign convention, whereby a positive value means gain and a negative one means cost; the choice between the two is a matter of taste and convenience only. The question of which finite games are representable using only nonnegative cost functions is not considered in this paper. However, see Section 4.2.

[5] In certain contexts (see Milchtaich 2012) it may be desirable to require the following (weak) connection between the players' weights and the cardinality of their strategy sets: For all i and j with $w_i < w_j$, $|S_i| \geq |S_j|$. In this paper, this requirement is not needed, but adding it would not affect any of the paper's results.

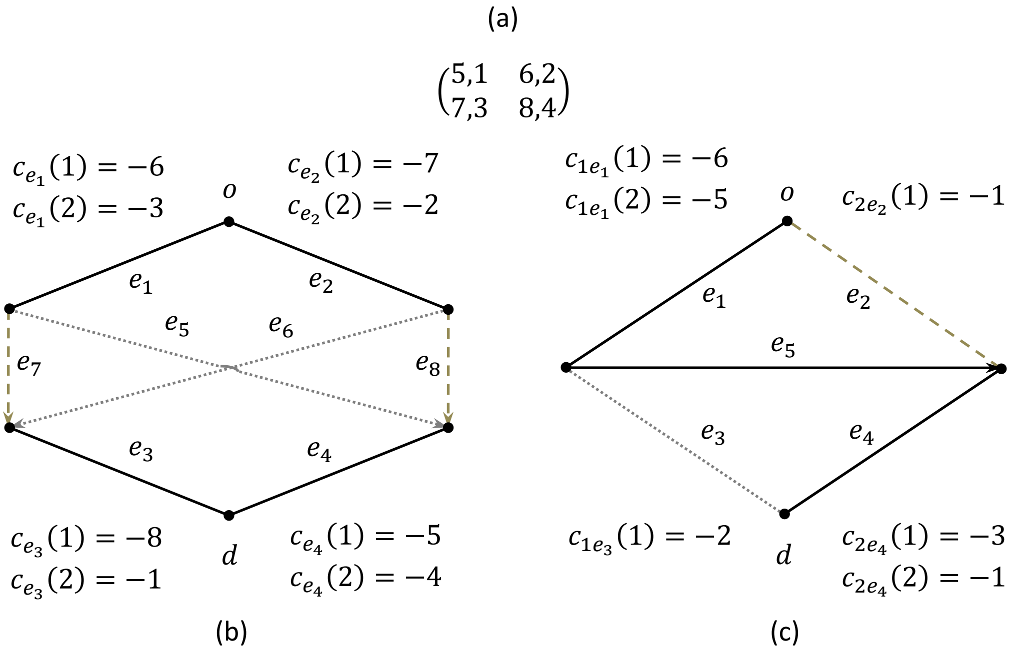

Fig. 1. Representations of a 2×2 exact potential game (a) as a weighted network congestion game (b), with weights $w_1 = 1$ and $w_2 = 2$, and as a network congestion game with player-specific costs (c). Dotted, dashed and solid edges are allowable to player 1, player 2 and both players, respectively. The allowable directions are indicated where needed. All relevant costs other than those specified are zero. A player's payoff is the negative of his total cost.

Theorem 1. Every finite game Γ is isomorphic both to a weighted network congestion game Γ' and to an (unweighted) network congestion game with player-specific costs Γ''. Γ is isomorphic to an unweighted network congestion game[6] if and only if it is an exact potential game.

By the first part of the theorem, every finite game can be represented as a network congestion game where the players differ in their weights and as one where they differ in their costs functions. The existence of a representation of the latter kind was first indicated by Monderer (2007). However, Monderer's definition of network congestion game is different, and significantly less restrictive, than in this paper. The second part of Theorem 1 strengths a well-known result of Monderer and Shapley (1996). In the following extension of that part of the theorem, their result is the equivalence of properties (ii) and (iii).

Theorem 2. For every finite game Γ, the following conditions are equivalent:

(i) Γ is isomorphic to an unweighted network congestion game.
(ii) Γ is isomorphic to a congestion game.
(iii) Γ is an exact potential game.

A finite game Γ obviously has more than a single pair of representations as in Theorem 1. The "canonical" games Γ' and Γ'' constructed in the proof of the theorem, which share the same network G, are just one such pair. Other representations may be preferable in that they use a simpler network. An example is shown in Fig. 1c. That representation of the 2×2

[6] This condition can be expressed as the requirement that $\Gamma' = \Gamma''$.

exact potential game in Fig. 1a as a network congestion game with player-specific costs uses the Wheatstone network, which has fewer edges than the network G constructed in the proof of Theorem 1 (which, for all 2×2 games, is the one in Fig. 1b). Both networks are considerably simpler than the network constructed in the proof of Theorem 2. For an example of a simple representation of a specific variety of congestion games as weighted network congestion games, see Milchtaich (2009).

Proof of Theorem 1. Suppose that the number n of players in Γ and the cardinality m of the largest strategy set are both at least two (otherwise the assertion is trivial). Without loss of generality, it may be assumed that the players are numbered in such a way that every player has at least as many strategies as every higher-number player. It is, however, desirable to have the stronger properly that the numbers of strategies are actually equal: *all* players have m strategies. To achieve such equality, (temporarily) increase the number of strategies of some players i by replicating one of their strategies. By definition, choosing the original strategy or any of its replicas has the same effect on i's payoff and on the payoffs of the other players. Each player's strategies can now be numbered from 1 to m. This numbering identifies the collection of all strategy profiles with the product set $\{1,2, \dots, m\}^n$. Order this set in the following way:

$$\begin{gathered}(1,1,\dots,1),(2,2,\dots,2),\dots,(m-1,m-1,\dots,m-1),(m,m,\dots,m),\\ \dots,(2,1,\dots,1),(3,2,\dots,2),\dots,(m,m-1,\dots,m-1),(1,m,\dots,m),\end{gathered} \tag{1}$$

where the order of the $m^n - 2m$ elements represented by the middle ellipsis mark is immaterial. With each strategy profile $s = (s_1, s_2, \dots, s_n)$ associate two vertices, u_s and v_s, and an edge e_s that joins them, is directed from u_s to v_s and is public. Next, for each player i and $1 \le k \le m$, consider all strategy profile s with $s_i = k$ and list them according to their order in (1). For each pair of successive entries in this list, s and t, add an edge that joins v_s and u_t, is directed from v_s to u_t, and is allowable only to player i. Finally, identify all vertices of the form u_s, where s is one of the first m elements in (1), and denote this single vertex by o. Do the same for all vertices of the form v_s, where s is one of the last m elements in (1), and denote the result by d. These terminal vertices, together with the other vertices and edges specified above, constitute a network G, with specified direction and set of allowable users for each edge. (For 2×2 games, this is the network depicted in Fig. 1b.) For each player i, each allowable route r in G corresponds to a unique strategy s_i in Γ. Specifically, r includes all m^{n-1} edges e_t with $t_i = s_i$, alternating with $m^{n-1} - 1$ private edges. Different allowable routes to player i have no shared edges, and no shared vertices other than the terminal ones.

The network congestion game with player-specific costs Γ'' is defined by assigning the following cost functions to each player i. For a public edge e_s, corresponding to a strategy profile s in Γ,

$$c_{ie_s}(x) = \begin{cases} 0, & x \le n-1 \\ K_i - h_i(s), & x = n \end{cases}, \tag{2}$$

where h_i is player i's payoff function in Γ and K_i is any number equal to or greater than the maximum of h_i. For a private edge e allowable only to player i, $c_{ie} = -K_i/(\,m^{n-1} - 1\,)$. As explained above, allowable route choices in G are in a one-to-one correspondence with strategy profiles in Γ. The n routes that correspond to a strategy profile s have exactly one common edge, namely, e_s. Therefore, for each player i, only e_s and $m^{n-1}-1$ of the private edges make a nonzero contribution to the cost. By (2), the total cost is $-h_i(s)$. Hence, player i's payoff is $h_i(s)$, the same payoff he receives in Γ.

The weighted network congestion game Γ' is defined by, first, attaching the weight $w_i = i + n - 2$ to each player i $(= 1,2,\dots,n)$. Thus, the weight uniquely identifies the player, and is less than the total weight of any two players, which is $2n-1$ or greater. Second, the cost functions are defined as follows. For a public edge e_s, corresponding to a strategy profile $s = (s_1, s_2, \dots, s_n)$ in Γ,

$$c_{e_s}(i+n-2) = \sum_{t_{-i}\in S_{-i}} \left(\prod_{j\neq i} \left(\frac{1}{m-1} - 1_{t_j=s_j} \right) \right) \big(K_i - h_i(s_i, t_{-i})\big), \quad i = 1,2,\dots,n \qquad (3)$$

and $c_{e_s}(x) = 0$ for $x \geq 2n-1$, where $K_1, K_2, \dots, K_n$ are any n numbers that make the cost functions of all public edges nondecreasing.[7] In this definition, S_{-i} is the collection of all partial strategy profiles $t_{-i} = (t_1, t_2, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$, the notation (s_i, t_{-i}) refers to the strategy profile $(t_1, t_2, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)$, and $1_{t_j=s_j}$ is defined as 1 if the indicated equality holds and 0 otherwise. For a private edge e, allowable only to a single player i, $c_e = -K_i/(\,m^{n-1}-1\,)$.

Consider the n routes in G that correspond to a particular strategy profile $s = (s_1, s_2, \dots, s_n)$ in Γ. The total cost of the private edges in the route of any player i is $-K_i$. The total cost of the public edges, which wholly comes from edges where player i is the sole user, can be computed as follows (to enhance readability, the computation is shown for $i = 1$):

$$\begin{aligned}
&\sum_{\bar{s}_2\neq s_2}\sum_{\bar{s}_3\neq s_3}\cdots\sum_{\bar{s}_n\neq s_n}\sum_{t_{-1}\in S_{-1}} \left(\prod_{j=2}^{n} \left(\frac{1}{m-1} - 1_{t_j=\bar{s}_j} \right) \right) \big(K_1 - h_1(s_1, t_{-1})\big) \\
&\qquad = \sum_{t_{-1}\in S_{-1}} \left(\prod_{j=2}^{n} \sum_{\bar{s}_j\neq s_j} \left(\frac{1}{m-1} - 1_{t_j=\bar{s}_j} \right) \right) \big(K_1 - h_1(s_1, t_{-1})\big) \\
&\qquad = \sum_{t_{-1}\in S_{-1}} \left(\prod_{j=2}^{n} 1_{t_j=s_j} \right) \big(K_1 - h_1(s_1, t_{-1})\big) \\
&\qquad = K_1 - h_1(s).
\end{aligned}$$

Therefore, also in Γ', a player's total cost is the negative of the corresponding payoff in Γ.

To complete the proof of the first part of the theorem it only remains to dispose of any spurious strategies that were introduced by replication at the beginning. These strategies can be eliminated simply by deleting or disallowing the use of all the private edges that belong to the corresponding routes. The second part of the theorem is covered by the proof of Theorem 2. ■

[7] For sufficiently large $K \geq 0$, the last requirement is satisfied by $K_i = (i-n-1)K$ (≤ 0), $i = 1,2,\dots,n$. This is because the coefficient of K_i in (3) is equal to the constant $1/(m-1)^{n-1}$.

Proof of Theorem 2. The fact that every congestion game, and in particular every unweighted network congestion game, is an exact potential game is well known (Monderer and Shapley 1996, Rosenthal 1973). The reverse implications are proved below, using a variant of the proof in Monderer and Shapley (1996).

Suppose that Γ has an exact potential P. As in the proof of Theorem 1, and for similar reasons, it is sufficient to consider the case in which the number n of players is at least 2 and the strategy set S_i of each player i includes at least two elements. (To achieve this, some players' strategies may need to be replicated, and the replicas disposed of at the end of the proof. Note that it would be possible in this way to also temporarily equalize the players' numbers of strategies. However, such equality would not add much to the proof.) Strategy profiles in Γ are naturally identifiable with subsets of the disjoint union $\coprod_{i=1}^{n} S_i$ (where 'disjoint' means that strategies of different players are viewed as distinct elements): $s = (s_1, s_2, \dots, s_n)$ is identified with the set $\{s_1, s_2, \dots, s_n\}$. Similarly, a partial strategy profile s_{-i}, which is obtained from a strategy profile s by ignoring the coordinate corresponding to a particular player i, is identified below with the set $\{s_1, s_2, \dots, s_n\} \cup S_i$. Player i (who, because of the assumption that each player has at least two strategies, is the only player j with $S_j \subseteq s_{-i}$) is said to *own* s_{-i}.

A congestion game isomorphic to Γ is defined as follows. All strategy profiles in Γ and all partial strategy profile are viewed as resources. (Thus, a resource is identified with a specific subset of $\coprod_{i=1}^{n} S_i$.) For each player i and strategy s_i of that player in Γ, the corresponding strategy in the congestion game is the selection of all the resources that include s_i. These resources are of three kinds: all strategy profiles containing s_i, all partial strategy profiles containing s_i but owned by another player, and all partial strategy profiles owned by i. The resources of the first two kinds are included only in the strategy in question but those of the third kind are shared by all of player i's strategies in the congestion game. It follows that player i is the sole user of (the resource identified with) a partial strategy profile if and only if (a) he owns it, i.e., the partial strategy profile is of the form s_{-i}, and (b) the strategy of every other player j is not (that corresponding to) his strategy s_j in s_{-i}. The cost for a player of using a resource, which is also the negative of the resource's contribution to the player's payoff, is defined as follows. For (the resource identified with) a strategy profile s, the cost is $-P(s)$ if all n players use the resource and 0 otherwise. For a partial strategy profile s_{-i} owned by some player i, the cost is

$$\frac{1}{|S_i|} \sum_{t \in S} \left(\prod_{j \neq i} \left(\frac{1}{|S_j| - 1} - 1_{t_j = s_j} \right) \right) \left(P(t) - h_i(t) \right) \quad (4)$$

if no one else uses the resource (which, as indicated, is possible only if the player using the resource is i himself) and 0 otherwise. In (4), h_i is player i's payoff function in Γ, S is the collection of all strategy profiles in that game, and $1_{t_j = s_j}$ is defined as 1 if the indicated equality holds and 0 otherwise. Note that a sufficient condition for the cost functions to be nondecreasing is that P and (4) are both nonpositive. Without loss of generality, this condition may be assumed to hold. Indeed, it is not difficult to see that subtracting any number c from P leaves it an exact potential function, and subtracts a certain constant fraction (specifically, $1/\prod_{j\neq i}(|S_j|-1)$) of c from (4).

By construction, strategy profiles in the congestion game are in a one-to-one correspondence with strategy profiles in Γ. To prove that the two games are isomorphic it remains to show that, for every strategy profile s, the total cost in the congestion game for every player i is $-h_i(s)$, the negative of his payoff in Γ. Two kinds of resources make a nonzero contribution to player i's cost: the single resource corresponding to the strategy profile s, which contributes $-P(s)$, and the resources corresponding to partial strategy profiles $\bar{s}_{-i}$ owned by i in which the strategy $\bar{s}_j$ of every player $j \neq i$ is *not* s_j. Assuming, for readability, that $i = 1$, the total contribution of the latter is

$$\begin{aligned}
\sum_{\bar{s}_2\neq s_2}\sum_{\bar{s}_3\neq s_3}\cdots\sum_{\bar{s}_n\neq s_n}\frac{1}{|S_1|}\sum_{t\in S}\left(\prod_{j=2}^{n}\left(\frac{1}{|S_j|-1}-1_{t_j=\bar{s}_j}\right)\right)\big(P(t)-h_1(t)\big)& \\
=\frac{1}{|S_1|}\sum_{t\in S}\left(\prod_{j=2}^{n}\sum_{\bar{s}_j\neq s_j}\left(\frac{1}{|S_j|-1}-1_{t_j=\bar{s}_j}\right)\right)\big(P(t)-h_1(t)\big)& \\
=\frac{1}{|S_1|}\sum_{t\in S}\left(\prod_{j=2}^{n}1_{t_j=s_j}\right)\big(P(s_1,t_{-1})-h_1(s_1,t_{-1})\big)& \\
=P(s)-h_1(s).&
\end{aligned}$$

(The second equality uses the fact that, by definition of exact potential, for every strategy profile t the difference $P(t)-h_1(t)$ does not depend player 1's strategy t_1.) This completes the proof of the isomorphism between Γ and the congestion game, which corresponds to the implication (iii) ⇒ (ii).

The above proof also constitutes the main step in the proof of the stronger assertion (iii) ⇒ (i). To complete the latter, the (unweighted) congestion game defined above needs to be turned into an unweighted network congestion game. To this end, the resources need to be viewed as public edges, which are supplemented and connected with a certain number of zero-cost private edges in the following manner. First, a single-route network is created for every strategy of every player i by connecting all the public edges that i uses only in that strategy (that is, resources of the first and second kinds considered above) alternately in series with private edges for that player, such that the first and last edges in the route are private. (The order of the public edges is immaterial.) Second, the resulting $|S_i|$ single-route networks are connected in parallel, and then connected in series with a similar single-route network in which all the public edges that i uses in all of his strategies (that is, resources of the third kind above) alternate with private edges. The final step is to identify the origin vertices of the n resulting networks (one network for each player) and similarly for their destination vertices. This gives an unweighted network congestion game that is trivially isomorphic to the congestion game defined above (and hence to Γ). This is because, for each strategy of each player i in the congestion game, all the resources that are included in that strategy, and only them, are also included in one of i's $|S_i|$ allowable routes in the network.

■

4 Variations

4.1 Acyclicity

An allowable route for a player by definition traverses each edge at most once. Relaxing this condition by allowing a higher number of repetitions can potentially enlarge the players' strategy sets by adding to them strategies that include cycles. There are two cases in which such a relaxation would not mean much. The first case, which is further discussed below, is that of nonnegative cost functions. With nonnegative costs, a strategy that includes a cycle is weakly dominated by the strategy obtained from it by removing the cycle. However, dominated strategies still appear in the game's normal form, so that the above relaxation is still not innocuous. The second case, in which the relaxation is totally inconsequential, in that in which cycles simply do not exist. That is, for every player i, starting at any vertex in the network, it is not possible to walk away and back to it by traversing (in the allowable directions) only edges that are allowable to i. This acyclicity condition holds for all the network congestion games specifically considered in this paper, and in particular for those in the proofs. Therefore, adding the condition as part of the definitions of the network congestion games would not affect any of the representation results.

4.2 Nonnegative Costs

In the proof of Theorem 2, for each player i all strategies in the congestion game include the same number of resources, and therefore all allowable routes in the network congestion game include the same number of edges. This entails that adding any constant K to all cost functions is strategically inconsequential in that it modifies each player's payoff function only by shifting it by some constant, which can moreover be made player-independent. The significance of this fact is that choosing K sufficiently large makes all cost functions nonnegative as well as nondecreasing. A similar remark applies to Theorem 1, which shows that, even with nonnegative cost functions, weighted network congestion games and network congestion games with player-specific costs can "almost" represent every finite game, specifically, represent it up to an additive constant.

4.3 Games Without Self-Effect

A third kind of network congestion games capable of representing all finite games is weighted network congestion games (in the wide sense) *without self-effect* (Milchtaich 2012). These games are similar to "normal" weighted network congestion games in that a single nondecreasing cost function d_e is associated with each edge e, which together with the flow f_e determines the cost of e for all users. However, the cost of e for a user i is given by $d_e(f_e - w_i)$, and it thus depends only on the total weight of the *other* users of e. Consequently, for users of different weights the cost may not be the same. In this respect, weighted network congestion games without self-effect are similar to network congestion games with player-specific costs. Using arguments similar to those given in the proof of Theorem 1, it is not difficult to show that every finite game can be represented as such a network congestion game.

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