The Equilibrium Existence Problem in Finite Network Congestion Games

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Abstract. An open problem is presented regarding the existence of pure strategy Nash equilibrium (PNE) in network congestion games with a finite number of non-identical players, in which the strategy set of each player is the collection of all paths in a given network that link the player's origin and destination vertices, and congestion increases the costs of edges. A network congestion game in which the players differ only in their origin–destination pairs is a potential game, which implies that, regardless of the exact functional form of the cost functions, it has a PNE. A PNE does not necessarily exist if (i) the dependence of the cost of each edge on the number of users is player- as well as edge-specific or (ii) the (possibly, edge-specific) cost is the same for all players but it is a function (not of the number but) of the total weight of the players using the edge, with each player *i* having a different weight w_i. In a parallel two-terminal network, in which the origin and the destination are the only vertices different edges have in common, a PNE always exists even if the players differ in either their cost functions or weights, but not in both. However, for general two-terminal networks this is not so. The problem is to characterize the class of all two-terminal network topologies for which the existence of a PNE is guaranteed even with player-specific costs, and the corresponding class for player-specific weights. Some progress in solving this problem is reported.

Keywords: Congestion games, network topology, heterogeneous users, existence of equilibrium

1 Introduction

1.1 Background

The theoretical study of congestion in networks began in the 1950's, at which time it was concerned mostly with transportation networks. The traffic flow was postulated to be at a so-called Wardrop equilibrium [30], in which the travel time on all used routes is equal, and less than or equal to that of a single vehicle on any unused route. An important milestone was the publication of Beckmann et al.'s book [3], which (under certain simplifying assumptions) presented the equilibrium as the optimal solution of a certain convex programming problem. In these authors' setting, users are nonatomic in the sense that the effect of any single user on the others is negligible.

Congestion games with a finite number of players, each with a non-negligible effect on the others, were first presented by Rosenthal [24]. He constructed what is now called an exact potential function on the space of strategy profiles and showed that every maximum point of the potential is a pure strategy Nash equilibrium (PNE) in the game. This is because, whenever a single player changes strategy, the change in that player's payoff is equal to the change in the potential function. Monderer and Shapley [19] showed that Rosenthal's games are in fact the only finite games for which an exact potential exists. Thus, any (finite) potential game can be presented as a congestion game, in which there is a finite set of common facilities and the strategy space of each player consists of subsets of facilities. The payoff from using each facility *j* depends only on the number of players whose chosen subset includes *j*. A special case of this is a network congestion game, in which the facilities correspond to the edges of a graph; the strategy space of each player is the collection of all directed paths, or routes, connecting two distinguished vertices, the player's origin and destination vertices; and the cost, or disutility, of using each edge is determined as a nondecreasing function by the flow on the edge. In Rosenthal's setting, players may differ only in their origin or destination vertices. If they are (i) differently effected by congestion, that is, have different cost functions, or (ii) have different weights, or congestion impacts, then the game is generally not a potential game and hence not a congestion game in Rosenthal's sense. For example, with player-specific costs, best-response cycles can occur if there are at least three players and at least three edges in the network [1,15]. Such cycles cannot occur in a potential game. Nevertheless, a network congestion game with either player-specific costs or weights, but not both, is guaranteed to have a PNE in the important special case of a parallel two-terminal network, i.e., one in which all players have the same origin–destination pair (in other words, a single-commodity network), which are the only vertices any two edges have in common [15]. In the case of player-specific weights, the result holds even if the weights are also edge-specific ("unrelated machines" [6]), and more generally, if the cost of each edge is an arbitrary nondecreasing function of the *set* of players using it [7]. A PNE does not necessarily exist, even in a parallel network, if the players have both player-specific costs and weights or if they are *positively* affected by congestion and the effects are player-specific [12,15,16].

The topological restriction on the network cannot be dispensed with. Libman and Orda [14] (see also [8]) gave an example of a two-terminal network with six edges for which there is a network congestion game with two players, one with twice the *weight* of the other, which does not have a PNE. They raised as an interesting subject for further research the problem of identifying non-parallel networks in which this is not possible, adding that series-parallel networks can be especially interesting. Konishi [11] gave an example of a different two-terminal network for which there is a three-player network congestion game with player-specific *costs* that does not have a PNE. He noted the similarity between the topological conditions for the existence of PNE and those for the *uniqueness* of the equilibrium in network congestion games with a *continuum* of non-identical players. (For such nonatomic games, existence of equilibrium is not an issue, since it is guaranteed by very weak assumptions on the cost functions [29].) Specifically, a parallel network is a sufficient condition in both cases.

The *equilibrium existence problem* that these authors point to is the identification of all two-terminal networks with the *topological existence property*: for any nondecreasing cost functions, with player-specific costs or weights (but not both), at least one PNE exists. This problem is substantially different from that of identifying classes of *cost functions* for which a PNE exists for all network topologies. An example of such a class is linear (more precisely, affine) functions. Regardless of the network topology, if all the players have identical, linear cost functions, a PNE always exists even with player-specific weights [8]. A similar distinction between the influences of the network topology and of the functional form of the cost functions applies also to the properties of efficiency and uniqueness, which are described below.

Efficiency of the equilibrium in a network congestion game has more than one possible meaning. It may refer to Pareto efficiency, that is, the impossibility of altering the players' route choices in a way that benefits them all, or to some aggregate measure of performance, such as the total cost or the cost of the worst route. In the latter case, the ratio between the chosen measure of performance at the worst Nash equilibrium and that at the social optimum is called the coordination ratio [13]. In nonatomic network congestion games with identical players, this ratio can be arbitrarily large for general cost functions, but it is bounded for certain families of functions, e.g., linear ones [28]. The least upper bound, dubbed the price of anarchy [22], is virtually independent of the network topology [25]. By contrast, the Pareto efficiency of the equilibria in nonatomic congestion games strongly depends on the topology. For a two-terminal network *G*, the equilibria are *always* Pareto efficient if and only if *G* has *linearly independent routes*, meaning that each route has an edge that is not in any other route [18]. In a sense, equilibria that are not Pareto efficient may occur in only three known two-terminal "forbidden" networks, which are the minimal ones *without* linearly independent routes. These results hold both with identical players and with player-specific cost functions.

For network congestion games with an arbitrary but finite number of players, who have identical cost functions but possibly different weights, the network topology is still irrelevant for the price of anarchy if the players may split their flow among multiple routes [26]. However, if the flow is unsplittable and only pure strategies are allowed [27], the (so-called *pure*) price of anarchy for linear cost functions apparently does depend on the network topology [2]. It also depends on whether or not the weights are player-specific [4]. (In the weighted case, the pure price of anarchy only refers to games in which a PNE exists.) The topological conditions for Pareto efficiency of the equilibrium in the unsplittable, pure-strategy case were found by Holzman and Law-yone [10]. These conditions are very similar to those applying to nonatomic network congestion games if the players are identical. However, if the players have different cost functions, there are virtually no topological conditions that guarantee Pareto efficiency: Pareto inefficient (and non-unique) equilibria occur in all two-terminal networks with at least two routes.

The problem of the *topological uniqueness* of the equilibrium is relevant for nonatomic network congestion games in which different players may have different cost functions. (With identical players, the equilibrium is always essentially unique. With a finite number of non-identical players and unsplittable flow, it is virtually impossible to guarantee uniqueness.) The class of all two-terminal networks for which uniqueness is guaranteed is defined by five simple kinds of networks, called the *nearly par-*

allel networks [17]. The complementary class of all two-terminal networks for which player-specific costs can result in multiple equilibria consists of all the networks in which one of four known "forbidden" networks is embedded. These results can be extended to network congestion games with finitely many players and splittable flow [23].

1.2 Results

This paper presents some partial results pertaining to the equilibrium existence problem, which is to identify the topological conditions guaranteeing the existence of at least one PNE in every network congestion game with player-specific costs or weights. The class of two-terminal network topologies for which the existence of a PNE is guaranteed is extended in a nontrivial manner beyond parallel networks. On the other hand, several new topologies are presented for which a PNE does not always exist. These results narrow the search for the problem's solution.

2 The Model

A *two-terminal network* (*network*, for short) G is defined in this work as a directed graph together with a distinguished pair of distinct vertices, the *origin* o and *destination* d, such that each vertex and each edge belong to at least one (directed) path $r = e_1 e_2 \cdots e_m$ linking o and d. Such a path is called a *route*. By definition, the terminal vertex of each edge e_j in a path except for the last one coincides with the initial vertex of the next edge, and all the vertices (and necessarily all the edges) are distinct [5]. This implies that loops are not allowed in G.[1] However, multiple edges are allowed.

For a given network G, a (finite) *network congestion game* is an n-player game, with $n \geq 1$, in which the strategy set of each player is the *route set* $\mathcal{R}$ of G, which consists of all the routes in the network. A *strategy profile* specifies a particular choice of route for each player. Players may differ from each other in their weight or cost functions.[2] The *weight* $w_i > 0$ of a player i is a measure of i's congestion impact. For an edge e in G, the total weight of the players whose routes include e, denoted by f_e, is the *flow* (or load) on e. The flow affects the *cost* of traversing e, which, for each player i, is given by a nonnegative, nondecreasing *cost function* $c_e^i : [0, \infty) \to [0, \infty)$. Thus, if the flow on e is f_e, its cost for i is $c_e^i(f_e)$. If the players have identical cost functions, this notation may be simplified to $c_e(f_e)$. If they all have the same weight, it may be assumed without loss of generality that the weight is 1. The cost of each route in the network for a player is the sum of the costs of its edges. The player's payoff in the game is the negative of this cost.

[1] Not allowing loops and other edges that do not belong to any route essentially involves no loss of generality, since such edges either cannot possibly be used or can only make a user's way unnecessarily long.

[2] *Multi-commodity networks*, in which players may also have different origin–destination pairs, and hence different strategy sets, are not considered in this paper.

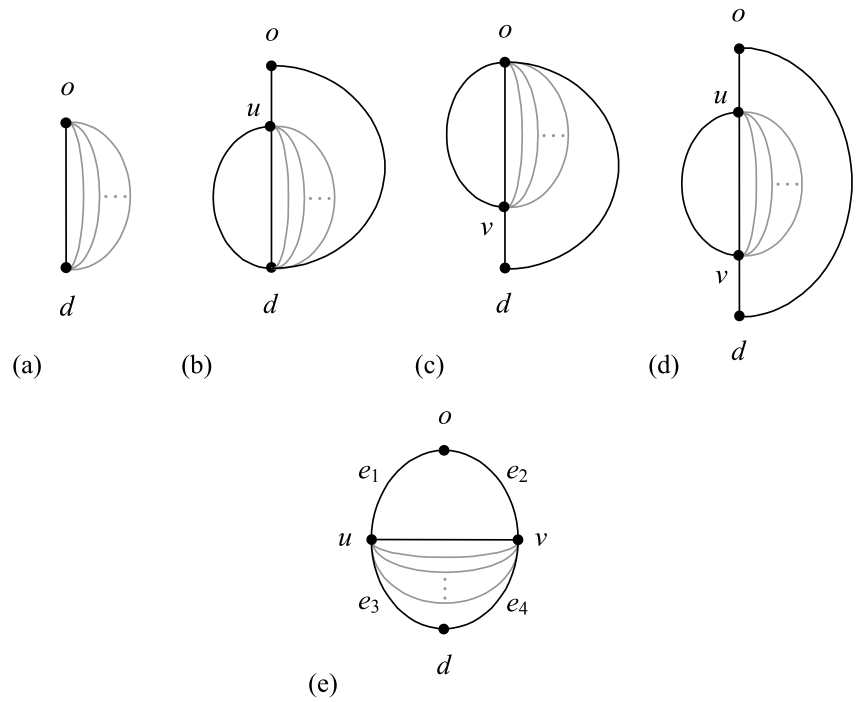

Fig. 1. Two-terminal networks with the topological existence property. In each network, the possible routes are the paths linking the origin *o* and destination *d*. Gray curves indicate optional edges. The directions of all the edges are unambiguous, except of those joining *u* and *v* in (e), which are assumed to be directed from *u* to *v*.

The following game theoretic terminology, which is not all standard, is used in this paper. A strategy profile is a *pure-strategy Nash equilibrium* (PNE) if none of the players can increase his payoff by unilaterally shifting to some other strategy. In other words, in a PNE each player plays a *best response* to the other players' strategies. The *superposition* of a finite number *m* of games with the same set of players is the game in which each of these players has to choose a strategy in each of the *m* games and his payoff is the sum of those in the *m* games [20]. Thus, the games are played simultaneously, but independently. Clearly, a strategy profile in the superposition of *m* games is a PNE if and only if it induces a PNE in each of these games. Two games Γ and Γ' with the same set of players and the same strategy set are *similar* if each player's payoff function in Γ is obtained from that in Γ' by adding to (or subtracting from) the latter a payoff function that depends only on the strategies of the other players. Similarity implies that the gain or loss for a player from unilaterally shifting from one strategy to another is the same in both games. Hence, it also implies that the games are *best-response equivalent*, i.e., a player's strategy is a best response to the others' strategies in one game if and only if this is so in the other game. Therefore, similar games have identical sets of PNEs.

3 Existence of PNE

If the players in a network congestion game differ in both their weights and cost functions, a PNE does not necessarily exist even in the case of a *parallel network* (Fig. 1(a)), which consists of one or more edges connected in parallel [15]. Therefore, for the notion of topological existence to be non-trivial, it is necessary to restrict attention to games in which players differ in only one of these respects. Doing so leads to the following positive result.

Theorem 1 [15]**.** *If G is a parallel network, then every network congestion game with player-specific costs or weights (but not both) has at least one PNE.*

For later reference, we note that this result can be slightly extended. Suppose that for each edge e there is pair of cost functions c_e and d_e (not necessarily different from zero) such that, for all players i and $x \geq w_i$,

$$c_e^i(x) = c_e(x) + d_e(x - w_i)\ . \tag{1}$$

Because of the second term in (1), if the players have different weights, they differ also in their cost functions. That term represents the effect of the other players using edge e on i; unlike the first term, it does not involve self-effect. Theorem 1 remains true if games with cost functions as in (1) are allowed. Such games will be referred to as network congestion games with player-specific weights *in the wide sense*. The existence of a PNE in this case can be proved by using the following algorithm (called greedy best response [9]). Players enter the game one after the other, ordered according to their weights from the highest to the lowest. Each player i chooses a route that is a best response to the route choices of the preceding players. It is not difficult to see that i's route remains a best response also after each of the remaining players i' enters the game, because $w_{i'} \leq w_i$ and the cost functions are nondecreasing. Therefore, the players' route choices constitute a PNE.

This constructive proof is specific to parallel networks; it cannot be extended in a straightforward manner to other network topologies. The same is true for all the other known proofs of Theorem 1, both for the case of player-specific costs and for player-specific weights [6,14,15]. In this respect, these proofs differ from that for the existence of PNE in network congestion games with identical players, for which the topology is irrelevant. Implicitly or explicitly, the latter proof uses the fact that every network congestion game Γ with identical players is similar (see the definition of similarity in Section 2) to some game Γ′ in which the players have identical payoff functions, i.e., their payoffs are always the same [19,21,24]. This argument does not extend to network congestion games with player-specific costs. Even for parallel networks, such games are generally not similar, or even best-response equivalent, to games with identical payoffs. Indeed, best-response cycles may occur [15].

Nevertheless, Theorem 1 can be extended to other network topologies. An immediate extension is to allow the connection of several parallel networks in series. In this case, by Theorem 1, the "restriction" of every network congestion game with player-specific costs or weights to any of the constituent parallel networks has a PNE. As the following lemma shows, this implies that the game itself has a PNE, since it is the superposition of these "restricted" games (see Section 2).

Lemma 1. *If a network G can be obtained by connecting a finite number m of networks in series, then every network congestion game Γ is the superposition of m network congestion games, each of which is obtained by considering the edges in only one constituent network. If each of these games has a PNE, then so does Γ.*

Proof. This follows immediately from the fact that each route in G is the concatenation of m paths, each of which is a route in one constituent network, and conversely, every such concatenation constitutes a route in G, whose cost for each player is the sum of the costs of its m parts. ■

Less obviously, Theorem 1 can be extended to networks that are not even series-parallel, such as the Wheatstone network (Fig. 1(e)). This extension is based on the following result.

Lemma 2. *For each of the networks G in Fig. 1 there is a parallel network $\tilde{G}$ such that, for every network congestion game Γ for G with player-specific costs or player-specific weights in the wide sense, there is a network congestion game $\tilde{\Gamma}$ for $\tilde{G}$ with the same property that is similar to Γ.*

Proof. Suppose, first, that G is as in Fig. 1(e). Let $\tilde{G}$ be the parallel network obtained from G by *contracting* e_1 and e_4, that is, replacing each of these edges and its two end vertices with a single vertex [5]. There is a natural one-to-one correspondence between the route sets of G and $\tilde{G}$, which allows network congestion games for these two networks to be viewed as having the same strategy set. For a given network congestion game Γ for G, with weights (w_i) and cost functions (c_e^i), let $\tilde{\Gamma}$ be the game for $\tilde{G}$ with the same weights (w_i) and the cost functions $(\tilde{c}_e^i)$ defined as follows: If $e = e_2$, then $\tilde{c}_e^i(x) = c_{e_2}^i(x) - c_{e_1}^i(w_i + w - x) + c$ for all i, where $w = \sum_i w_i$ is the players' total weight and c is an arbitrary large constant (which serves to make the cost nonnegative). If $e = e_3$, then $\tilde{c}_e^i(x) = c_{e_3}^i(x) - c_{e_4}^i(w_i + w - x) + c$. Finally, if $e \neq e_2, e_3$, then $\tilde{c}_e^i(x) = c_e^i(x) + c$. If Γ is a game with player-specific costs but identical weights, then $\tilde{\Gamma}$ clearly has the same property. The same is true if Γ is a game with player-specific weights in the wide sense, since (1) implies that $\tilde{c}_{e_2}^i(x)$, for example, can be written as $c_{e_2}(x) - d_{e_1}(w - x) + c/2 + d_{e_2}(x - w_i) - c_{e_1}(w - (x - w_i)) + c/2$.

It remains to show that the games Γ and $\tilde{\Gamma}$ are similar. That is, for every choice of routes by the players and every player i, the difference between the cost in Γ and that in $\tilde{\Gamma}$ depends only on the routes of the other players. If i's route does not include e_2 or e_3, this difference is

$$c_{e_1}^i(w_i + w'_{-i}) + c_{e_4}^i(w_i + w''_{-i}) - c\ , \qquad (2)$$

where w'_{-i} is the total weight of the players other than i whose route does not include e_2, and w''_{-i} is the corresponding weight for e_3. The same expression gives the difference between the costs in Γ and $\tilde{\Gamma}$ also if i's route includes either e_2 or e_3. Thus, the difference is independent of i's route, as had to be shown.

The above argument can easily be adapted for each of the other networks in Fig. 1. For networks G as in Fig. 1(b) and (c), $\tilde{G}$ is obtained by contracting only one edge. The network in (d) can be reduced to either of the previous two by moving one of the edges incident with the terminal vertices so that it becomes adjacent with the other such edge. Clearly, this rearrangement of edges does not affect the cost of any route.

Alternatively, the validity of the conclusion of the lemma for each of the other networks in Fig. 1 can easily be deduced from that for (e). ■

The assertion of Lemma 2 cannot be strengthened to *identity*, or isomorphism, between Γ and $\tilde{\Gamma}$. If the costs in Γ are player-specific, it may be qualitatively different from all network congestion games with the same property for parallel networks. For example, whereas games of the latter kind are always sequentially solvable [16], there are examples showing that Γ does not necessarily have this property. However, for present purposes, similarity is more than sufficient, since it implies that every PNE in $\tilde{\Gamma}$ is also a PNE in Γ. By Theorem 1 and the remark following it, at least one such PNE exists. Together with Lemma 1, this gives the following.

Theorem 2. *If G is one of the networks in Fig. 1 or can be obtained by connecting several of these networks in series*, *then every network congestion game with player-specific costs or weights (but not both) has a PNE.*

It is not known whether Theorem 2 can be extended to include also networks similar to those in Fig. 1(e) but with the reverse directions for some of the edges joining u and v. These networks and those in Fig. 1 are the directed versions of the *nearly parallel* networks [17], which are essentially the only two-terminal networks for which uniqueness of the equilibrium in nonatomic network congestion games with player-specific costs is guaranteed. This adds weight to Konishi's [11] observation that the conditions for topological existence (for a finite number of players with different cost functions) are similar to the conditions for topological uniqueness (for a continuum of such players). However, Theorem 2 leaves open the question of whether for every two-terminal network that is not nearly parallel there is a network congestion game with player-specific costs that does not have a PNE. Some results concerning this question are presented below.

4 Non-existence of PNE

The network in Fig. 2(d), which is obtained by connecting the Wheatstone network (Fig. 1(e)) in parallel with a single edge, differs substantially from the former in that network congestion games with player-specific costs or weights do not necessarily have a PNE. An example showing this for the case of different weights was given by Libman and Orda [14], and another one by Fotakis et al. [8]. These two examples are very similar to each other and to the next one; the different examples differ only in the cost functions.

Example 1. Two players simultaneously choose routes in the network in Fig. 2(d). The players have different weights, $w_1 = 1$ and $w_2 = 2$, but the same cost functions, given by $c_{e_1}(x) = 4x + 16$, $c_{e_2}(x) = 45$, $c_{e_3}(x) = 48$, $c_{e_4}(x) = x^3 - 9x^2 + 28x$, $c_{e_5}(x) = 16x$ and $c_{e_6}(x) = 65x$. For player 2, using e_5 is never optimal, since its cost is at least 32 whereas the difference between the costs of e_2 and e_1 is always less than that. Using e_6 is also never optimal for 2, since its cost is at least 130, which is always greater than $c_{e_1} + c_{e_3}$. This leaves player 2 with only two routes to choose from, and implies that 1 is the only player who may use e_5. The cost of that edge for player 1 is therefore 16, which is always less than the difference between the costs of e_2 and e_1, as well as the difference between the costs of e_3 and e_4. Therefore, using e_2 or e_3 is never optimal for player 1, which leaves him with only two possible routes, $r_1 = e_6$ and $r_2 = e_1\, e_5\, e_4$. If player 1's route is r_1 or r_2, the best-response route for 2 is $r_3 = e_1\, e_3$ or $r_4 = e_2\, e_4$, respectively. However, if player 2 uses r_3 or r_4, the best response for 1 is r_2 or r_1, respectively. Therefore, a PNE does not exist. Note that this would be true also if the constant functions c_{e_2} and c_{e_3} were replaced by sufficiently slowly increasing linear ones. However, if (the nonlinear) c_{e_4} were replaced by a linear function, a PNE *would* exist [8].

Essentially the same example shows that, in the network in Fig. 2(d), existence of a PNE is not guaranteed also with player-specific costs. This network is simpler than (i.e., it is a subnetwork of) the one used in Konishi's [11] example.

Example 2. This example is similar to the previous one, expect that the players differ not in their weights, which are given by $w_1 = w_2 = 1$, but in their cost functions, which are derived from those in Example 1 in the following manner: For each edge e, $c_e^1(1) = c_e(1)$, $c_e^2(1) = c_e(2)$, and $c_e^1(2) = c_e^2(2) = c_e(3)$. Clearly, the two-player game thus defined is identical to that in Example 1, and hence it does not have a PNE.

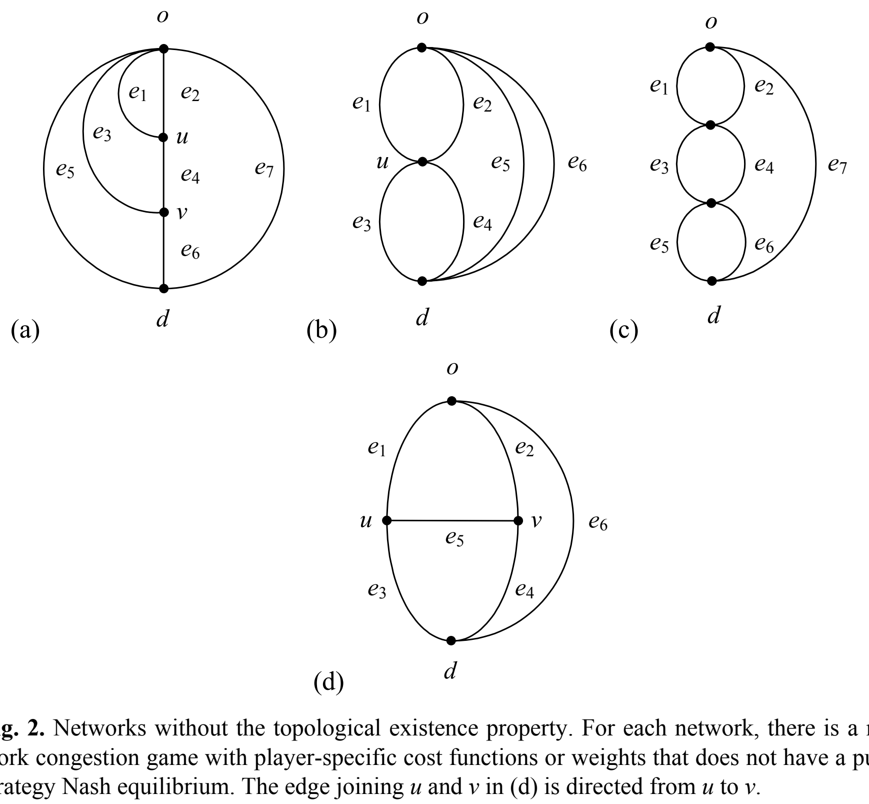

Fig. 2. Networks without the topological existence property. For each network, there is a network congestion game with player-specific cost functions or weights that does not have a pure-strategy Nash equilibrium. The edge joining u and v in (d) is directed from u to v.

Network congestion games without a PNE exist also for certain series-parallel networks. It is not known, however, whether these networks are the same for the cases of player-specific costs and player-specific weights. For the networks in Fig. 2(b) and (c), a PNE does not necessarily exist if the players have different cost functions. The next example concerns the former.

Example 3. Three players, all with weight 1, simultaneously choose routes in the network in Fig. 2(b). The cost of each edge for each player is given in Table 1. Effectively, each player has only one possible short route *s* (of length 1) and one long route *l* (of length 2). Player 1's *l* shares an edge (e_4) with 2's *l*, and his *s* coincides with 3's *s*. For player 1, the cost of *s* is less or greater than that of *l* if player 3 takes his *l* or *s*, respectively. Similarly, for player 2, *s* is preferable to *l* or the other way around if player 1 takes his *l* or *s*, respectively; and for 3, *s* is preferable to *l* or the other way around if player 2 takes his *l* or *s*, respectively. Clearly, this implies that a strategy profile in which everyone's route is optimal does not exist.

Table 1. Cost functions for Example 3. For each player, the cost of each edge as a function of the flow on it is shown. Blank cells indicate prohibitively high costs.

	e_1	e_2	e_3	e_4	e_5	e_6
Player 1		$3x$		$3x$		$5x$
Player 2	x			$3x$	$6x$	
Player 3	x		x			$x/3+2$

The network in the next example is not only series-parallel but is even ("extension-parallel" [10], or) a network with linearly independent routes [18].

Example 4. Three players, with weights $w_1 = 1$, $w_2 = 2$ and $w_3 = 4$, choose routes in the network in Fig. 2(a). The players' identical cost functions are given in Table 2. For player 3, there are effectively only two possible routes, $e_2\, e_4\, e_6$ and e_7. If the player chooses the former, then (regardless of what 1 does) player 2's best response is e_5, to which 1's best response is $e_1\, e_4\, e_6$. It is then better for player 3 to switch from $e_2\, e_4\, e_6$ (whose cost is 14) to e_7. However, if he chooses e_7, then (regardless of what 1 does) player 2's best response is $e_3\, e_6$, to which 1's best response is e_5. It is then better for player 3 to switch from e_7 to $e_2\, e_4\, e_6$ (whose cost is 12½). This proves that a PNE does not exist.

Table 2. Cost functions for Example 4. For each value of the flow on an edge, its cost (for all players) is shown. Blank cells indicate prohibitively high costs.

Flow	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	6	5	⅛	1	1	13
2		6	5½	¼	10	2	13
3		6	6	⅜	11	3	13
4		6		½		4	13
5				3		5	
6						6	
7						7	

These examples establish the possible non-existence of PNE also in many other networks, namely, those in which one or more of the four networks in Fig. 2 is embedded. For example, adding edges to any of the four networks would not make any difference, since extra edges can be effectively eliminated by assigning a very high cost to them. "Embedding" is used here in a somewhat generic sense. There are at least two different meanings for this term that may be relevant in the present context [17,18]. Very roughly, they correspond to the notions of a minor and topological minor of a graph [5].

Many two-terminal networks other than those mentioned above exist. Solving the equilibrium existence problem entails placing each of them either in the class of networks for which the existence of a PNE is guaranteed or in the class of those for which a network congestion game without a PNE exists. Whether this partition is the same for games with player-specific cost functions and for player-specific weights is not known.

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