Congestion Games with Player-Specific Constants*

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Abstract. We consider a special case of *weighted congestion games* with *player-specific latency functions* where each player uses for each particular resource a fixed (non-decreasing) *delay function* together with a player-specific *constant*. For each particular resource, the resource-specific delay function and the player-specific constant (for that resource) are composed by means of a *group operation* (such as addition or multiplication) into a player-specific latency function. We assume that the underlying group is a *totally ordered abelian group*. In this way, we obtain the class of *weighted congestion games with player-specific constants*; we observe that this class is contained in the new intuitive class of *dominance weighted congestion games*. We obtain the following results:

Games on parallel links:

- Every unweighted congestion game has a *generalized ordinal potential*.
- There is a weighted congestion game with 3 players on 3 parallel links that does not have the *Finite Best-Improvement Property*.
- There is a particular *best-improvement cycle* for general weighted congestion games with player-specific latency functions and 3 players whose outlaw implies the existence of a pure Nash equilibrium. This cycle is indeed outlawed for dominance weighted congestion games with 3 players – and hence for weighted congestion games with player-specific constants and 3 players.

Network congestion games:
For unweighted *symmetric network congestion games* with player-specific *additive* constants, it is $\mathcal{PLS}$-complete to find a pure Nash equilibrium.

Arbitrary (non-network) congestion games:
Every weighted congestion game with *linear* delay functions and player-specific additive constants has a *weighted potential*.

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1 Introduction

Motivation and Framework. Originally introduced by Rosenthal [15], *congestion games* model resource sharing among *(unweighted) players*. Here, the strategy of each player is a set of *resources*. The cost for a player on resource e is given by a *latency function* for e, which depends on the total weight of all players choosing e. In *congestion games with player-specific latency functions*, which were later introduced by Milchtaich [13], players are *weighted* and each player chooses her own latency function for each resource, which determines her own player-specific cost on the resource. These choices reflect different preferences, beliefs or estimates by the players; for example, such differences occur in multiclass networks or in networks with uncertain parameters.

In this work, we introduce a special case of (weighted) congestion games with player-specific latency functions [13], which we call *(weighted) congestion games with player-specific constants*. Here, each player-specific latency function is made up of a resource-specific *delay function* and a player-specific *constant* (for the particular resource); the two are composed by means of a *group* operation. We will be assuming that the underlying group is a *totally ordered abelian group* (see, for example, [9, Chapter 1]). Note that this new model of congestion games restricts Milchtaich's one [13] since player-specific latency functions are no longer completely arbitrary; simultaneously, it generalizes the weighted generalization of Rosenthal's model [15] since it allows composing player-specific constants into each (resource-specific) latency function. For example, *(weighted) congestion games with player-specific additive constants* (resp., *multiplicative constants*) correspond to the case where the group operation is addition (resp., multiplication).

We will sometimes focus on *network congestion games*, where the resources and strategies correspond to edges and paths in a (directed) *network*, respectively; network congestion games offer an appropriate model for some aspects of *routing* problems. In such games, each player has a *source* and a *destination* node and her strategy set is the set of all paths connecting them. In a *symmetric* network congestion game, all players use the same pair of source and destination; else, the network congestion game is *asymmetric*. The simplest symmetric network congestion game is the *parallel links* network with only two nodes.

The *Individual Cost* for a player is the sum of her costs on the resources in her strategy. In a *(pure) Nash equilibrium*, no player can decrease her Individual Cost by unilaterally deviating to a different strategy. We shall study questions of existence of, computational complexity of, and convergence to pure Nash equilibria for (weighted) congestion games with player-specific constants.

For convergence, we shall consider sequences of *improvement* and *best-improvement* steps of players; in such steps, a player *improves* (that is, decreases) and *best-improves* her Individual Cost, respectively. A game has the *Finite Improvement Property* [14] (resp., the *Finite Best-Improvement Property*, also called *Finite Best-Reply Property* [13]) if all *improvement paths* (resp., *best-improvement paths*) are finite. Both properties imply the existence of a pure Nash equilibrium [14]; clearly, the first property implies the second. Also, the existence of a *generalized ordinal potential* is equivalent to the Finite Improvement Property [14] (and hence it implies the Finite Best-Improvement Property and the existence of a pure Nash equilibrium as well). A *weighted potential*

[14] is a particular case of a generalized ordinal potential; an *exact potential* [14] is a particular case of a weighted potential.

We observe that the class of (weighted) congestion games with player-specific constants is contained in the more general, intuitive class of *dominance (weighted) congestion games* that we introduce (Proposition 1). In this more general class, it holds that for any pair of players, the preference of some of the two players with regard to any arbitrary pair of resources necessarily induces an identical preference for the other player (Definition 2).

State-of-the-Art. It is known that every *unweighted* congestion game has a pure Nash equilibrium [15]; Rosenthal's original proof uses an *exact potential* [14]. It is possible to compute a pure Nash equilibrium for an unweighted symmetric network congestion game in polynomial time by reduction to the *min-cost flow problem* [3]. However, the problem becomes $\mathcal{PLS}$-complete for either (arbitrary) symmetric congestion games [3] or asymmetric network congestion games where the edges of the network are either directed [3] or undirected [1]. Weighted asymmetric network congestion games with affine latency functions are known to have a pure Nash equilibrium [6]; in contrast, there are weighted symmetric network congestion games with non-affine latency functions that have no pure Nash equilibrium (even if there are only 2 players) [6,12]. Weighted (network) congestion games on parallel links have the Finite Improvement Property (and hence a pure Nash equilibrium) if all latency functions are non-decreasing; in this setting, [5] proves that a pure Nash equilibrium can be computed in polynomial time by using the classical LPT algorithm due to Graham [10] when latency functions are linear. (This is the well-known setting of *related parallel links*, which is equivalent to using the identity function for all delay functions in a weighted congestion game with multiplicative constants.) In the general case, it is strongly $\mathcal{NP}$-complete to determine whether a given weighted network congestion game has a pure Nash equilibrium [2].

For weighted congestion games with (non-decreasing) player-specific latency functions on parallel links, there is a counterexample to the existence of a pure Nash equilibrium with only 3 players and 3 links [13]. This result is *tight* since such games with 2 players have the Finite Best-Improvement Property [13]. Unweighted congestion games with (non-decreasing) player-specific latency functions have a pure Nash equilibrium but not necessarily the Finite Best-Improvement Property [13].

The special case of (weighted) congestion games with player-specific *linear* latency functions (without a constant term) was studied in [7,8]. Such games have the Finite Improvement Property if players are unweighted [7], while there is a game with 3 weighted players that does not have it [7]. For the case of 3 weighted players, every congestion game with player-specific linear latency functions (without a constant term) has a pure Nash equilibrium but not necessarily an exact potential [8]. For the case of 2 links, there is a polynomial time algorithm to compute a pure Nash equilibrium [8]. A larger class of (incomplete information) unweighted congestion games with player-specific latency functions that have the Finite Improvement Property has been identified in [4]; the special case of our model where the player-specific constants are *additive* is contained in this larger class.

Contribution and Significance. We partition our results on congestion games with player-specific constants according to the structure of the strategy sets in the congestion game:

Games on parallel links:

- Every unweighted congestion game with player-specific constants has a generalized ordinal potential (Theorem 1). (Hence, every such game has the Finite Improvement Property and a pure Nash equilibrium.) The proof employs a potential function involving the group operation; the proof that this function is a generalized ordinal potential explicitly uses the assumption that the underlying group is a totally ordered abelian group. We remark that Theorem 1 does *not* need the assumption that the (resource-specific) delay functions are non-decreasing.

  Theorem 1 simultaneously broadens two corresponding state-of-the-art results for two very special cases: *(i)* each delay function is the identity function and the group operation is *multiplication* [7] and *(ii)* the group operation is addition [4]. We note that, in fact, the potential function we used is a generalization of the potential function used in [4] (for addition) to an arbitrary group operation. However, [4] applies to *all* unweighted congestion games.
- It is *not* possible to generalize Theorem 1 to weighted congestion games (with player-specific constants): there is such a game with 3 players on 3 parallel links that does not have the Finite Best-Improvement Property – hence, neither the Finite Improvement Property (Theorem 2). To prove this, we provide a simple counterexample for the case of player-specific additive constants.
- Note that Theorem 2 does not outlaw the possibility that every weighted congestion game with player-specific constants has a pure Nash equilibrium. Although we do not know the answer for the general case with an arbitrary number of players, we have settled the case with 3 players: every weighted congestion game with player-specific constants and 3 players has a pure Nash equilibrium (Corollary 3). The proof proceeds in two steps.

  First, we establish that there is a particular best-improvement cycle whose outlaw implies the existence of a pure Nash equilibrium (Theorem 3). We remark that an identical cycle had been earlier constructed by Milchtaich for the more general class of weighted congestion games with player-specific latency functions [13, Section 8].

  Second, we establish that this particular best-improvement cycle is indeed outlawed for the more specific class of dominance weighted congestion games (Theorem 4). Since a weighted congestion game with player-specific constants is a dominance weighted congestion game, the cycle is outlawed for weighted congestion games with player-specific constants as well; this implies the existence of a pure Nash equilibrium for them (Corollary 3). This implies, in particular, a separation of this specific class from the general class of congestion games with player-specific latency functions with respect to best-improvement cycles.

  We remark that Corollary 3 broadens the earlier result by Georgiou *et al.* [8, Lemma B.1] for congestion games with player-specific multiplicative constants and identity delay functions.

Network congestion games:

Recall that every unweighted congestion game with player-specific additive constants has a pure Nash equilibrium [4]. Nevertheless, we establish that it is $\mathcal{PLS}$-complete to compute one (Theorem 5) even for a symmetric network congestion game. The proof uses a simple reduction from the $\mathcal{PLS}$-complete problem of computing a pure Nash equilibrium for an unweighted asymmetric network congestion game [3].

Arbitrary (non-network) congestion games:

Note that Theorem 2 outlaws the possibility that every weighted congestion game with player-specific constants has the Finite Best-Improvement Property. Nevertheless, we establish that every weighted congestion game with player-specific constants has a weighted potential for the special case of linear delay functions and player-specific additive constants (Theorem 6). (Hence, every such game has the Finite Improvement Property and a pure Nash equilibrium).

The proof employs a potential function and establishes that it is a weighted potential. For the special case of weighted asymmetric network congestion games with affine latency functions (which are not player-specific), the potential function we used reduces to the potential function introduced in [6] for the corresponding case.

Theorems 1 and 3 suggest that the class of congestion games with player-specific constants provides a vehicle for reaching the limit of the existence of potential functions towards the direction of player-specific costs.

2 Framework and Preliminaries

Totally Ordered Abelian Groups. A *group* $(G, \odot)$ consists of a *ground set* G together with a binary operation $\odot : G \times G \rightarrow G$; $\odot$ is associative and allows for an *identity element* and *inverses*. The group $(G, \odot)$ is *abelian* if $\odot$ is commutative. We will consider *totally ordered abelian groups* with a *total order* on G [9] which satisfies *translation invariance*: for all triples $r, s, t \in G$, if $r \leq s$ then $r \odot t \leq s \odot t$. Examples of totally ordered abelian groups include (i) $(\mathbb{R}_{>0}, \cdot)$ under the usual number-ordering, and (ii) $(\mathbb{R}^2, +)$ under the *lexicographic ordering* on pairs of numbers. We will often focus on the case where G is $\mathbb{R}$ (the set of reals).

Congestion Games. For all integers $k \geq 1$, we denote $[k] = \{1, \ldots, k\}$. A *weighted congestion game with player-specific latency functions* [13] is a tuple $\Gamma = (n, E, (w_i)_{i\in[n]}, (S_i)_{i\in[n]}, (f_{ie})_{i\in[n],e\in E})$. Here, n is the number of *players* and E is a finite set of *resources*. For each player $i \in [n]$, $w_i > 0$ is the *weight* and $S_i \subseteq 2^E$ is the *strategy set* of player i. For each pair of player $i \in [n]$ and resource $e \in E$, $f_{ie} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is a non-decreasing player-specific *latency function*. In the *unweighted* case, $w_i = 1$ for all players $i \in [n]$.

In a (weighted) *network congestion game* (with player-specific latency functions), resources and strategies correspond to edges and paths in a directed network. In such games, each player has a *source* and a *destination* node, each of her strategies is a path from source to destination and all paths are possible. In a *symmetric* network congestion game, all players use the same pair of source and destination; else, the network congestion game is *asymmetric*. In the *parallel links* network, there are only two nodes; this gives rise to symmetric network congestion games.

Definition 1. *Fix a totally ordered abelian group* $(G, \odot)$. *A* ***weighted congestion game with player-specific constants*** *is a weighted congestion game* Γ *with player-specific latency functions such that (i) for each resource* $e \in E$*, there is a non-decreasing* delay function $g_e : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$*, and (ii) for each pair of a player* $i \in [n]$ *and a resource* $e \in E$*, there is a player-specific constant* $c_{ie} > 0$*, so that for each player* $i \in [n]$ *and resource* $e \in E$*,* $f_{ie} = c_{ie} \odot g_e$*.*

In a *weighted congestion game with player-specific additive constants* (resp., *player-specific multiplicative constants*), G is $\mathbb{R}$ and $\odot$ is $+$ (resp., G is $\mathbb{R}_{>0}$ and $\odot$ is $\cdot$). The special case of weighted congestion games with player-specific constants where for all players $i \in [n]$ and resources $e \in E$, $c_{ie} = \epsilon$ (the *identity element* of G) yields the *weighted congestion games* generalizing the unweighted congestion games introduced by Rosenthal [15]. So, (weighted) congestion games with player-specific constants fall between the weighted generalization of congestion games [15] and (weighted) congestion games with player-specific latency functions [13].

We now prove that, in fact, congestion games with player-specific constants are contained within a more restricted class of congestion games with player-specific latency functions that we introduce. Fix a weighted congestion game Γ with player-specific latency functions. Consider a pair of (distinct) players $i, j \in [n]$ and a pair of (distinct) resources $e, e' \in E$. Say that i *dominates* j for the ordered pair $\langle e, e' \rangle$ if for every pair of positive numbers $x, y \in \mathbb{R}_{>0}$, $f_{ie}(x) > f_{ie'}(y)$ implies $f_{je}(x) > f_{je'}(y)$. Intuitively, i dominates j for $\langle e, e' \rangle$ if the decision of i to switch her strategy from e to e' always implies a corresponding decision for j; in other words, j always follows the decision of i (to switch or not) for the pair $\langle e, e' \rangle$.

Definition 2. *A weighted congestion game with player-specific latency functions is a* ***dominance (weighted) congestion game*** *if for all pairs of players* $i, j \in [n]$*, for all pairs of resources* $e, e' \in E$*, either* i *dominates* j *for* $\langle e, e' \rangle$ *or* j *dominates* i *for* $\langle e, e' \rangle$*.*

We prove:

Proposition 1. *A (weighted) congestion game with player-specific constants is a dominance (weighted) congestion game.*

Proof. Fix a pair of players $i, j \in [n]$ and a pair of resources $e, e' \in E$. We proceed by case analysis. Assume first that $c_{ie} \odot c_{je'} \geq c_{ie'} \odot c_{je}$. We will show that j dominates i for $\langle e, e' \rangle$. Fix a pair of numbers $x, y \in \mathbb{R}_{>0}$. Assume that $f_{je}(x) > f_{je'}(y)$ or $c_{je} \odot g_e(x) > c_{je'} \odot g_{e'}(y)$. By translation-invariance, it follows that $c_{ie} \odot c_{je} \odot g_e(x) > c_{ie} \odot c_{je'} \odot g_{e'}(y)$. The assumption that $c_{ie} \odot c_{je'} \geq c_{ie'} \odot c_{je}$ implies that $c_{ie} \odot c_{je'} \odot g_{e'}(y) \geq c_{ie'} \odot c_{je} \odot g_{e'}(y)$. It follows that $c_{ie} \odot g_e(x) > c_{ie'} \odot g_{e'}(y)$ or $f_{ie}(x) > f_{ie'}(y)$. Hence, j dominates i for $\langle e, e' \rangle$.

Assume now that $c_{ie'} \odot c_{je} > c_{ie} \odot c_{je'}$. We will show that i dominates j for $\langle e, e' \rangle$. Fix a pair of numbers $x, y \in \mathbb{R}_{>0}$. Assume that $f_{ie}(x) > f_{ie'}(y)$ or $c_{ie} \odot g_e(x) > c_{ie'} \odot g_{e'}(y)$. By translation-invariance, it follows that $c_{je} \odot c_{ie} \odot g_e(x) > c_{je} \odot c_{ie'} \odot g_{e'}(y)$. The assumption that $c_{ie'} \odot c_{je} > c_{ie} \odot c_{je'}$ implies that $c_{je} \odot c_{ie'} \odot g_{e'}(y) > c_{je'} \odot c_{ie} \odot g_{e'}(y)$. It follows that $c_{je} \odot g_e(x) > c_{je'} \odot g_{e'}(y)$ or $f_{je}(x) > f_{je'}(y)$. Hence, i dominates j for $\langle e, e' \rangle$. $\square$

Profiles and Individual Cost. A *strategy* for player $i \in [n]$ is some specific $s_i \in S_i$. A *profile* is a tuple $\mathsf{s} = (s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$. For the profile s, the *load* $\delta_e(\mathsf{s})$ on resource $e \in E$ is given by $\delta_e(\mathsf{s}) = \sum_{i \in [n] \mid s_i \ni e} w_i$. For the profile s, the *Individual Cost* of player $i \in [n]$ is given by $\mathsf{IC}_i(\mathsf{s}) = \sum_{e \in s_i} f_{ie}(\delta_e(\mathsf{s})) = \sum_{e \in s_i} c_{ie} \odot g_e(\delta_e(\mathsf{s}))$.

Pure Nash Equilibria. Fix a profile s. A player $i \in [n]$ is *satisfied* if she cannot decrease her Individual Cost by unilaterally changing to a different strategy; else, player i is *unsatisfied*. So, an unsatisfied player i can take an *improvement step* to decrease her Individual Cost; if player i is satisfied after the improvement step, the improvement step is called a *best-improvement step*. An *improvement cycle* (resp., *best-improvement cycle*) is a cyclic sequence of improvement steps (resp., best-improvement steps). A game has the *Finite Improvement Property* (resp., *Finite Best-Improvement Property*) if all sequences of improvement steps (resp., best-improvement steps) are finite; clearly, the Finite Improvement Property (resp., the Finite Best-Improvement Property) outlaws improvement cycles (resp., best-improvement cycles). Clearly, the Finite Improvement Property implies the Finite Best-Improvement Property. A profile is a *(pure) Nash equilibrium* if all players are satisfied. Clearly, the Finite Improvement Property implies the existence of a pure Nash equilibrium (as also does the Finite Best-Improvement Property), but not vice versa [14].

A *generalized ordinal potential* for the game Γ [14] is a function $\Phi : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$ that decreases when a player takes an improvement step. Say that a function $\Phi : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$ is a *weighted potential* for the game Γ [14] if there is a *weight vector* $\mathsf{b} = (b_i)_{i \in [n]}$ such that for every player $k \in [n]$, for every profile s, and for every strategy $t_k \in S_k$ that transforms s to t, it holds that $\mathsf{IC}_k(\mathsf{s}) - \mathsf{IC}_k(\mathsf{t}) = b_k \cdot (\Phi(\mathsf{s}) - \Phi(\mathsf{t}))$. If this even holds for the vector b with $b_i = 1$ for all $i \in [n]$, the function Φ is an *exact potential* [14]. A game has a generalized ordinal potential if and only if it has the Finite Improvement Property (and hence the Finite Best-Improvement Property and a pure Nash equilibrium) [14].

$\mathcal{PLS}$(-complete) Problems. $\mathcal{PLS}$ [11] includes optimization problems where the goal is to find a *local optimum* for a given instance; this is a feasible solution with no feasible solution of better objective value in its well-determined neighborhood. A problem Π in $\mathcal{PLS}$ has an associated set of instances $\mathcal{I}_\Pi$. There is, for every instance $I \in \mathcal{I}_\Pi$, a set of feasible solutions $\mathcal{F}(I)$. Furthermore, there are three polynomial time algorithms A, B and C. A computes for every instance I a feasible solution $S \in \mathcal{F}(I)$; B computes for a feasible solution $S \in \mathcal{F}(I)$, the objectice value; C determines, for a feasible solution $S \in \mathcal{F}(I)$, whether S is locally optimal and, if not, it outputs a feasible solution in the neighborhood of S with better objective value.

A $\mathcal{PLS}$-problem Π_1 is *$\mathcal{PLS}$-reducible* [11] to a $\mathcal{PLS}$-problem Π_2 if there are two polynomial time computable functions F_1 and F_2 such that F_1 maps instances $I \in \mathcal{I}_{\Pi_1}$ to instances $F_1(I) \in \mathcal{I}_{\Pi_2}$ and F_2 maps every local optimum of the instance $F_1(I)$ to a local optimum of I. A $\mathcal{PLS}$-problem Π is *$\mathcal{PLS}$-complete* [11] if every problem in $\mathcal{PLS}$ is $\mathcal{PLS}$-reducible to Π.

3 Congestion Games on Parallel Links

We now introduce a function $\Phi : S_1 \times \ldots \times S_n \to \mathbb{R}$ with

$$\Phi(\mathsf{s}) = \bigodot_{e \in E} \bigodot_{i=1}^{\delta_e(\mathsf{s})} g_e(i) \odot \bigodot_{i=1}^{n} c_{is_i}.$$

for any profile s. We prove that this function is a generalized ordinal potential:

Theorem 1. *Every unweighted congestion game with player-specific constants on parallel links has a generalized ordinal potential.*

Proof. Fix a profile s. Consider an improvement step of player $k \in [n]$ to strategy t_k, which transforms s to t. Clearly, $\mathsf{IC}_k(\mathsf{s}) > \mathsf{IC}_k(\mathsf{t})$ or

$$g_{s_k}(\delta_{s_k}(\mathsf{s})) \odot c_{ks_k} > g_{t_k}(\delta_{t_k}(\mathsf{t})) \odot c_{kt_k}.$$

Note also that $\delta_{s_k}(\mathsf{t}) = \delta_{s_k}(\mathsf{s}) - 1$ and $\delta_{t_k}(\mathsf{t}) = \delta_{t_k}(\mathsf{s}) + 1$, while $\delta_e(\mathsf{t}) = \delta_e(\mathsf{s})$ for all $e \in E \setminus \{s_k, t_k\}$. Hence,

$$\begin{aligned}
&\Phi(\mathsf{s}) \\
&= \bigodot_{e \in E \setminus \{s_k,t_k\}} \bigodot_{i=1}^{\delta_e(\mathsf{s})} g_e(i) \odot \bigodot_{i \in [n] \setminus \{k\}} c_{is_i} \odot \bigodot_{i=1}^{\delta_{s_k}(\mathsf{s})} g_{s_k}(i) \odot \bigodot_{i=1}^{\delta_{t_k}(\mathsf{s})} g_{t_k}(i) \odot c_{ks_k} \\
&= \bigodot_{\substack{e \in E \setminus \\ \{s_k,t_k\}}} \bigodot_{i=1}^{\delta_e(\mathsf{s})} g_e(i) \odot \bigodot_{\substack{i \in [n] \\ \setminus \{k\}}} c_{is_i} \odot \bigodot_{i=1}^{\delta_{s_k}(\mathsf{s})-1} g_{s_k}(i) \odot \bigodot_{i=1}^{\delta_{t_k}(\mathsf{s})} g_{t_k}(i) \odot g_{s_k}(\delta_{s_k}(\mathsf{s})) \odot c_{ks_k} \\
&> \bigodot_{\substack{e \in E \setminus \\ \{s_k,t_k\}}} \bigodot_{i=1}^{\delta_e(\mathsf{s})} g_e(i) \odot \bigodot_{\substack{i \in [n] \\ \setminus \{k\}}} c_{is_i} \odot \bigodot_{i=1}^{\delta_{s_k}(\mathsf{s})-1} g_{s_k}(i) \odot \bigodot_{i=1}^{\delta_{t_k}(\mathsf{s})} g_{t_k}(i) \odot g_{t_k}(\delta_{t_k}(\mathsf{t})) \odot c_{kt_k} \\
&= \bigodot_{e \in E \setminus \{s_k,t_k\}} \bigodot_{i=1}^{\delta_e(\mathsf{t})} g_e(i) \odot \bigodot_{i \in [n] \setminus \{k\}} c_{is_i} \odot \bigodot_{i=1}^{\delta_{s_k}(\mathsf{t})} g_{s_k}(i) \odot \bigodot_{i=1}^{\delta_{t_k}(\mathsf{t})} g_{t_k}(i) \odot c_{kt_k} \\
&= \Phi(\mathsf{t}),
\end{aligned}$$

so that Φ is a generalized ordinal potential. □

Theorem 1 immediately implies:

Corollary 1. *Every unweighted congestion game with player-specific constants on parallel links has the Finite Improvement Property and a pure Nash equilibrium.*

We continue to prove:

Theorem 2. *There is a weighted congestion game with additive player-specific constants and 3 players on 3 parallel links that does* not *have the Finite Best-Improvement Property.*

Proof. By construction. The weights of the 3 players are $w_1 = 2$, $w_2 = 1$, and $w_3 = 1$. The player-specific constants and resource-specific delay functions are as follows:

c_{ie}	Link 1	Link 2	Link 3
Player 1	0	∞	5
Player 2	0	0	∞
Player 3	∞	0	2

	Link 1	Link 2	Link 3
$g_e(1)$	1	2	1
$g_e(2)$	8	13	2
$g_e(3)$	14	∞	10

Notice that the profiles $\langle 1,2,3\rangle$ and $\langle 3,1,2\rangle$ are both Nash equilibria. Consider now the cycle $\langle 1,1,3\rangle \rightarrow \langle 1,1,2\rangle \rightarrow \langle 1,2,2\rangle \rightarrow \langle 3,2,2\rangle \rightarrow \langle 3,2,3\rangle \rightarrow \langle 3,1,3\rangle \rightarrow \langle 1,1,3\rangle$. The Individual Cost of the deviating player decreases in each of these steps:

	IC_1	IC_2	IC_3
$\langle 1,1,3\rangle$	14		3
$\langle 1,1,2\rangle$		14	2

	IC_1	IC_2	IC_3
$\langle 1,2,2\rangle$	8	13	
$\langle 3,2,2\rangle$	7		13

	IC_1	IC_2	IC_3
$\langle 3,2,3\rangle$		2	12
$\langle 3,1,3\rangle$	15	1	

So, this is an improvement cycle. Furthermore, note that each step in this cycle is a best-improvement step, so this is actually a best-improvement cycle. The claim follows. □

We continue to consider the special case of 3 players but for the general case of weighted congestion games with player-specific constants. We prove:

Theorem 3. *Let Γ be a weighted congestion game with player-specific latency functions and 3 players on parallel links. If Γ does* not *have a best-improvement cycle $\langle l,j,j\rangle \rightarrow \langle l,l,j\rangle \rightarrow \langle k,l,j\rangle \rightarrow \langle k,l,l\rangle \rightarrow \langle k,j,l\rangle \rightarrow \langle l,j,l\rangle \rightarrow \langle l,j,j\rangle$ (where $l \neq j$, $j \neq k$, $l \neq k$ are any three links and $w_1 \geq w_2 \geq w_3$), then Γ has a pure Nash equilibrium.*

We now continue to prove:

Theorem 4. *Every dominance weighted congestion game with 3 players on parallel links does not have an improvement cycle of the form $\langle l,j,j\rangle \rightarrow \langle l,l,j\rangle \rightarrow \langle k,l,j\rangle \rightarrow \langle k,l,l\rangle \rightarrow \langle k,j,l\rangle \rightarrow \langle l,j,l\rangle \rightarrow \langle l,j,j\rangle$ where $l \neq j$, $j \neq k$, $l \neq k$ are any three links and $w_1 \geq w_2 \geq w_3$.*

Proof. Assume, by way of contradiction, that there is a dominance congestion game with such a cycle. Since all steps in the cycle are improvement steps, one gets for player 2 that $f_{2j}(w_2 + w_3) > f_{2l}(w_1 + w_2)$ and $f_{2l}(w_2 + w_3) > f_{2j}(w_2)$. In the same way, one gets for player 3 that $f_{3j}(w_3) > f_{3l}(w_2 + w_3)$ and $f_{3l}(w_1 + w_3) > f_{3j}(w_2 + w_3)$. We proceed by case analysis on whether 2 dominates 3 or 3 dominates 2 for $\langle j,l\rangle$.

Assume first that 2 dominates 3 for $\langle j,l\rangle$. Then, the first inequality for player 2 implies that $f_{3j}(w_2+w_3) > f_{3l}(w_1+w_2) \geq f_{3l}(w_1+w_3)$ (since f_{3l} is non-decreasing and $w_2 \geq w_3$), a contradiction to the second inequality for player 3. Assume now that 3 dominates 2 for $\langle j,l\rangle$. Then, the first inequality for player 3 implies that $f_{2l}(w_2+w_3) < f_{2j}(w_3) \leq f_{2j}(w_2)$ (since f_{2j} is non-decreasing and $w_2 \geq w_3$), a contradiction to the second inequality for player 2. □

Since dominance (weighted) congestion games are a subclass of (weighted) congestion games with player-specific latency functions, Theorems 3 and 4 immediately imply:

Corollary 2. *Every dominance weighted congestion game with 3 players on parallel links has a pure Nash equilibrium.*

By Proposition 1, Corollary 2 immediately implies:

Corollary 3. *Every weighted congestion game with player-specific constants and 3 players on parallel links has a pure Nash equilibrium.*

4 Network Congestion Games

Theorem 5. *It is $\mathcal{PLS}$-complete to compute a pure Nash equilibrium in an unweighted symmetric network congestion game with player-specific additive constants.*

Proof. Clearly, the problem of computing a pure Nash equilibrium in an unweighted symmetric congestion game with player-specific additive constants is a $\mathcal{PLS}$-problem. (The set of feasible solutions is the set of all profiles and the neighborhood of a profile is the set of profiles that differ in the strategy of exactly one player; the objective function is the generalized ordinal potential since a local optimum of this functions is a Nash equilibrium [14].) To prove $\mathcal{PLS}$-hardness, we use a reduction from the $\mathcal{PLS}$-complete problem of computing a pure Nash equilibrium for an unweighted, asymmetric network congestion game [3]. For the reduction, we construct the two functions F_1 and F_2:

$\underline{F_1}$: Given an unweighted, asymmetric network congestion game Γ on a network G, where $(a_i, b_i)_{i\in[n]}$ are the source and destination nodes of the n players and $(f_e)_{e\in E}$ are the latency functions, F_1 constructs a symmetric network congestion game Γ' with n players on a graph G', as follows:

- G' includes G, where for each edge e of G, $g'_e := f_e$ and $c'_{ie} = 0$ for each $i \in [n]$.
- G' contains a new common source a' and a new common destination b'; for each player $i \in [n]$, we add an edge (a', a_i) with $g'_{(a',a_i)}(x) := 0$, $c'_{i(a',a_i)} := 0$, and $c'_{k(a',a_i)} := \infty$ for all $k \neq i$; in addition we add for each player $i \in [n]$ an edge (b_i, b') with $g'_{(b_i,b')}(x) := 0$, $c'_{i(b_i,b')} := 0$, and $c'_{k(b_i,b')} := \infty$ for all $k \neq i$.

$\underline{F_2}$: Consider now a pure Nash equilibrium t for Γ'. The function F_2 maps t to a profile s for Γ (which, we shall prove, is a Nash equilibrium for Γ) as follows:

- Note first that for each player $i \in [n]$, t_i (is a path that) includes both edges (a', a_i) and (b_i, b') (since otherwise $\mathsf{IC}_i(\mathsf{t}) = \infty$). Construct s_i from t_i by eliminating the edges (a', a_i) and (b_i, b').

It remains to prove that $\mathsf{s} = F_2(\mathsf{t})$ is a Nash equilibrium for Γ. By way of contradiction, assume otherwise. Then, there is a player k that can decrease her Individual Cost in Γ by changing her path s_k to s'_k. But then player k can decrease her Individual Cost in Γ' by changing her path $t_k = (a', a_k), s_k, (b_k, b')$ to $t'_k = (a', a_k), s'_k, (b_k, b')$. So, t is not a Nash equilibrium for Γ'. A contradiction. □

We remark that Theorem 5 holds also for unweighted symmetric network congestion games with player-specific additive constants and *undirected* edges since the problem of computing a pure Nash equilibrium for an unweighted, asymmetric network congestion game with undirected edges is also $\mathcal{PLS}$-complete [1].

5 Arbitrary Congestion Games

We now restrict attention to weighted congestion games with player-specific additive constants c_{ie} and linear delay functions $f_e(x) = a_e \cdot x$. This gives rise to *weighted congestion games with player-specific affine latency functions* $f_{ie}(x) = a_e \cdot x + c_{ie}$, where $i \in [n]$ and $e \in E$. For this case, we introduce a function $\Phi : S_1 \times \ldots \times S_n \to \mathbb{R}$ with $\Phi(\mathsf{s}) = \sum_{i=1}^{n} \sum_{e \in s_i} w_i \cdot (2 \cdot c_{ie} + a_e \cdot (\delta_e(s) + w_i))$, for any profile s. For any pair of player $i \in [n]$ and resource $e \in E$, define $\phi(\mathsf{s}, i, e) = w_i \cdot (2 \cdot c_{ie} + a_e \cdot (\delta_e(\mathsf{s}) + w_i))$, so that $\Phi(\mathsf{s}) = \sum_{i=1}^{n} \sum_{e \in s_i} \phi(\mathsf{s}, i, e)$. We now prove that this function is a weighted potential:

Theorem 6. *Every weighted congestion game with player-specific affine latency functions has a weighted potential.*

Proof. Fix a profile s. Assume that player $k \in [n]$ unilaterally changes to the strategy t_k, which transforms s to t. Clearly,

$$\begin{aligned}
&\Phi(\mathsf{s}) - \Phi(\mathsf{t}) \\
&= \sum_{i \in [n]} \sum_{e \in s_i} \phi(\mathsf{s}, i, e) - \sum_{i \in [n]} \sum_{e \in t_i} \phi(\mathsf{t}, i, e) \\
&= \sum_{e \in s_k} \phi(\mathsf{s}, k, e) - \sum_{e \in t_k} \phi(\mathsf{t}, k, e) + \sum_{i \in [n] \setminus \{k\}} \left(\sum_{e \in s_i} \phi(\mathsf{s}, i, e) - \sum_{e \in t_i} \phi(\mathsf{t}, i, e) \right)
\end{aligned}$$

We treat separately the first and the second part of this expression. On one hand,

$$\begin{aligned}
\sum_{e \in s_k} \phi(\mathsf{s}, k, e) - \sum_{e \in t_k} \phi(\mathsf{t}, k, e) &= \sum_{e \in s_k \setminus t_k} \phi(\mathsf{s}, k, e) - \sum_{e \in t_k \setminus s_k} \phi(\mathsf{t}, k, e) \\
= \sum_{e \in s_k \setminus t_k} w_k (2 \cdot c_{ke} + a_e \cdot (\delta_e(\mathsf{s}) + w_k)) &- \sum_{e \in t_k \setminus s_k} w_k (2 \cdot c_{ke} + a_e \cdot (\delta_e(\mathsf{t}) + w_k)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{i \in [n] \setminus \{k\}} \left(\sum_{e \in s_i} \phi(\mathsf{s}, i, e) - \sum_{e \in t_i = s_i} \phi(\mathsf{t}, i, e) \right) = \sum_{i \in [n] \setminus \{k\}} \sum_{e \in s_i} (\phi(\mathsf{s}, i, e) - \phi(\mathsf{t}, i, e)) \\
&= \sum_{\substack{i \in \\ [n] \setminus \{k\}}} \left(\sum_{e \in s_i \cap (s_k \setminus t_k)} (\phi(\mathsf{s}, i, e) - \phi(\mathsf{t}, i, e)) + \sum_{e \in s_i \cap (t_k \setminus s_k)} (\phi(\mathsf{s}, i, e) - \phi(\mathsf{t}, i, e)) \right) \\
&= \sum_{e \in s_k \setminus t_k} \sum_{\substack{i \in [n] \setminus \{k\} \\ \mid e \in s_i}} (\phi(\mathsf{s}, i, e) - \phi(\mathsf{t}, i, e)) + \sum_{e \in t_k \setminus s_k} \sum_{\substack{i \in [n] \setminus \{k\} \\ \mid e \in s_i}} (\phi(\mathsf{s}, i, e) - \phi(\mathsf{t}, i, e)) \\
&= \sum_{\substack{e \in \\ s_k \setminus t_k}} \sum_{\substack{i \in [n] \setminus \{k\} \\ \mid e \in s_i}} (w_i \cdot a_e \cdot (\delta_e(\mathsf{s}) - \delta_e(\mathsf{t}))) + \sum_{\substack{e \in \\ t_k \setminus s_k}} \sum_{\substack{i \in [n] \setminus \{k\} \\ \mid e \in s_i}} (w_i \cdot a_e \cdot (\delta_e(\mathsf{s}) - \delta_e(\mathsf{t}))) \\
&= w_k \cdot \sum_{e \in s_k \setminus t_k} a_e \cdot (\delta_e(\mathsf{s}) - w_k) - w_k \cdot \sum_{e \in t_k \setminus s_k} a_e \cdot (\delta_e(\mathsf{t}) - w_k) \,.
\end{aligned}$$

Putting these together yields that Φ is a weighted potential with weight vector b having $b_i = \frac{1}{2w_i}$, $i \in [n]$. □

Theorem 6 immediately implies:

Corollary 4. *Every weighted congestion game with player-specific affine latency functions has the Finite Improvement Property and a pure Nash equilibrium.*

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