GAMES WITH CONGESTION-AVERSE AND CONGESTION-SEEKING PLAYERS

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Abstract. The paper studies a class of resource-symmetric singleton congestion games with two types of players having diametrically opposite preferences. *Congestion-averse* players wish to avoid congestion while *congestion-seeking* players favor it. We show that a pure-strategy Nash equilibrium may or may not exist, depending on the number of players of each type and the number of resources in the game. The same numbers also determine whether the game is acyclic with respect to unilateral best-improvement moves, that is, whether such moves always lead to an equilibrium. We also study the sequential-move versions of the game, in which the players choose resources one by one after observing the choices of all preceding players, and cannot later change them. The players' choices in a subgame perfect equilibrium in this game do not necessary constitute an equilibrium in the original, simultaneous-move game. However, the converse does hold: every equilibrium in the simultaneous-move game is a sequential-move equilibrium, in the sense that it is obtained as the equilibrium path in some subgame perfect equilibrium for some entrance order.

Keywords: Congestion Games, Congestion-Seeking Players, Weak Acyclicity, Weak Potential, Sequential-Move Equilibrium, Pure-Strategy Equilibrium

1. INTRODUCTION

Congestion games are a class of non-cooperative games first introduced by Rosenthal (1973). In a congestion game, each strategy is a particular subset of a common set of resources. The utility associated with each resource is a function of the number of players who include it in their choice. Each player's payoff is the sum of the utilities associated with the resources included in his choice. Every game in this class has at least one pure-strategy Nash equilibrium. This result follows from the existence of an *exact potential* (Monderer and Shapley, 1996)—a real-valued function over the set of (pure) strategy profiles having the property that the gain or loss of a player shifting to a new strategy is equal to the corresponding increment of the potential function. The existence of a potential moreover implies that the game has the *finite improvement property*, or FIP (Monderer and Shapley, 1996): any sequence of strategy profiles in which each entry differs from the preceding one only in the strategy of a single player, whose deviation strictly increases the payoff he receives, is finite.

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Several variants and special cases of Rosenthal's congestion games, making different assumptions about the players and resources, have been studied. These include singleton congestion games, in which each strategy is a single resource, and games in which the utility functions, which specify the dependence of the payoff from using each resource on the number of its users, are player-specific (Milchtaich, 1996). Many recent papers assume monotone utility functions: either decreasing or, less commonly, increasing (Rozenfeld and Tennenholtz, 2006). In this paper, we assume that the game has both *congestion-averse players*, who prefer to share their resource with as few others as possible, and *congestion-seeking players*, whose utility from using a resource increases as its number of users increases. On the other hand, the resources in our model are all identical.

The presence of the two opposite kinds of players affects the existence of pure-strategy Nash equilibrium. Unlike the cases of only congestion-averse or congestion-seeking players, existence of equilibrium is not guaranteed even with identical resources. As a simple example, consider a game with two players of opposite types and two (identical) resources. There is no equilibrium, since the congestion-seeking player would like to share a resource with the congestion-averse one while the latter would prefer to avoid him. However, as we show in this paper, the players' opposite preferences are not totally incompatible with equilibrium existence. We identify all combinations of the numbers of players of each type and of resources for which an equilibrium does exist.

The existence of equilibrium raises the question of convergence to it, in particular, whether the game has the finite improvement property as in Rosenthal's games. In fact, it is not difficult to see that, with three or more resources, our games *never* have that property (see Section 4). However, in some cases they possess the weaker *finite best-improvement*, or *acyclicity*, *property.* This means that an improvement path is necessary finite if in each step the unique deviator shifts to a strategy that is a best response against the strategies played by the other players. For the case in which the game has an equilibrium but is not acyclic, we prove that from any initial strategy profile there is *some* best-improvement path that ends in an equilibrium. In other words, games with congestion-averse and congestion-seeking players are always *weakly acyclic.* These results are summarized in Table 1.

An improvement path represents myopic behavior by players. In each step, one player chooses a resource that maximizes his current payoff, but does not consider the effect of his choice on the others' future behavior. We also examine convergence to equilibrium for players who are "forward looking". Specifically, we study sequential-move versions of the game, in which players enter the game one by one in a particular order. An entering player chooses his resource after observing the choices of the preceding players, and after the entrance of the last player, the payoffs are determined according to the utility functions in the original, simultaneous-move game. In general, the equilibrium outcomes in a sequential-move version of a strategic game may be very different from those in the simultaneous-move game. For example, the equilibrium payoffs in a Cournot competition are different from those in the

The quotient $\frac{n-1}{m}$ is:	Property
An integer	*Acyclic* (Theorem 2)
A non-integer less than $n^s - 1$	*Weakly Acyclic* (Theorem 3)
Otherwise	*No Equilibrium* (Theorem 1)

Table 1. Whether a game with congestion-averse and congestion-seeking players possesses a pure-strategy Nash equilibrium, and whether the players may spontaneously converge to it, is completely determined by the number of players n, the number of congestion-seeking players n^s and the number of resources m. For each of the three cases on the left column, the right column gives the strongest property of the game.

corresponding Stackelberg model. We show, however, that if the entrance order is such that congestion-seeking players precede the congestion-averse ones, there exists a subgame perfect equilibrium whose equilibrium path is an equilibrium in the original, simultaneous-move game. Moreover, all the equilibria in that game can be obtained this way.

The next section formally describes our model. The equilibrium existence theorem is presented in Section 3. Convergence to equilibrium is examined in Section 4, where the proofs of the results in Table 1 are given. In Section 5 we explore the sequential-move versions of the game and the relations between their subgame perfect equilibria and the equilibria in the original game.

2. THE MODEL

A (singleton) *congestion model* with *congestion-averse* and *congestion-seeking players* is defined as follows. There are n players ($i = 1,2, \dots, n$), who must each choose one of m (≥ 2) identical common resources. The payoff of a player who chooses resource j ($j = 1,2, \dots, m$) is a function of the number n_j of players using that resource. Depending on the player, this *utility function* is either u^a or u^s. The first function is monotonically decreasing,

$$u^a(k) > u^a(k+1), \qquad 1 \leq k \leq n-1,$$

and the second function is monotonically increasing,

$$u^s(k) < u^s(k+1), \qquad 1 \leq k \leq n-1.$$

A player with utility function u^a is said to be *congestion-averse* and a player to whom u^s applies is *congestion-seeking*. We assume that the numbers n^a and n^s ($= n - n^a$) of players of each kind are both at least 1. Otherwise, the game would reduce to a special case of Rosenthal's classic model.

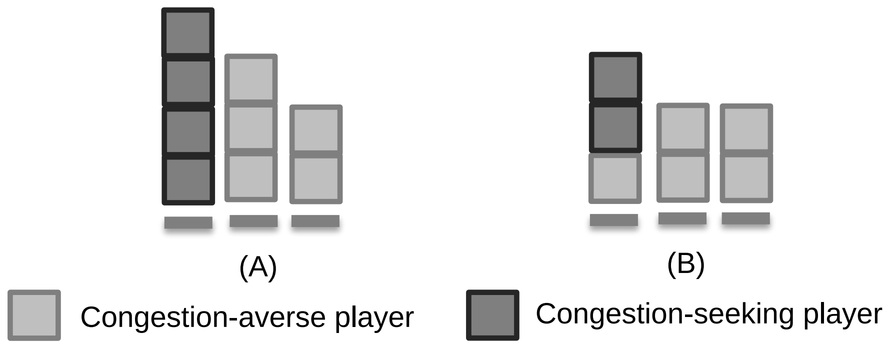

Figure 1. An equilibrium in (A) a game with many congestion-seeking players (condition (i) in Theorem 1 holds) and (B) a game with fewer such players (condition (ii) holds).

Note that in our model, unlike Rosenthal's one, all resources are identical. Consequently, the model would not have been more general if we allowed different congestion-averse or congestion-seeking players to have different utility functions. This is because allowing this would not change the preferences of each kind of player: a change of resource benefits a congestion-averse or congestion-seeking player if and only if the resource he moves to has fewer or more users, respectively, than the one he leaves.

A congestion model as above defines a (simultaneous-move) *congestion game* Γ, in which players choose resources simultaneously and receive their payoffs according to their utility functions. Together with a specified ordering of the players, it also defines a *sequential-move* version of Γ, which is the perfect-information extensive-form game in which the players choose resources one after the other according to the specified order rather than simultaneously. Whereas for the simultaneous-move game the basic solution concept we employ is pure-strategy Nash equilibrium, for the sequential-move one it is subgame perfect equilibrium.

3. EXISTENCE OF EQUILIBRIUM

As shown above, games with congestion-averse and congestion-seeking players do not generally admit a pure-strategy Nash equilibrium. The next theorem identifies necessary and sufficient condition for equilibrium existence.

Theorem 1. A game with congestion-averse and congestion-seeking players has a pure-strategy Nash equilibrium if and only if the quotient $\frac{n-1}{m}$ is (i) less than $n^s - 1$ or (ii) an integer.

Condition (i), $n^s > \frac{n-1}{m} + 1$, means that the number of congestion-seeking players in the game is relatively large. Condition (ii) means that division of the total number of players (of either type) by the number of resources leaves a remainder of one, that is, $n \equiv 1 \bmod m$. Figure 1 shows a typical equilibrium configuration for each case.

Proof. Suppose that $\frac{n-1}{m}$ satisfies condition (i). Let all congestion-seeking players be concentrated on the first resource and the congestion-averse players distributed as equally as possible on the $m-1$ other resources (which means they all have at least $\left\lfloor\frac{n-n^s}{m-1}\right\rfloor$ users but some of them may have one additional user; see Figure 1(A)). Since all congestion-seeking players use the resource with the largest number of users, none of them would benefit from changing resource. The congestion-averse players also do not have an incentive to move, since the number of users of any alternative resource is smaller by at most 1. Therefore, this configuration is an equilibrium.

Suppose now that condition (i) does not hold but (ii) holds. This means that the players can be distributed among the resources in such a way that $n_1 - 1 = n_2 = \cdots = n_m = \frac{n-1}{m}$. Considering that (i) does not hold, we can moreover place all congestion-seeking players on the first resource, which has the largest number of users. By the same argument used in the previous case, this configuration is an equilibrium.

Finally, suppose that both (i) and (ii) do not hold. We have to show that an equilibrium does not exist. In any equilibrium, all congestion-seeking players must use a resource that has a larger number of users than any other resource. Suppose this is resource 1, so that $n_1 - 1 \geq n_2, \dots, n_m$, and therefore $n_1 - 1 \geq \frac{n-1}{m}$. On the other hand, since by assumption both (i) and (ii) do not hold, $\frac{n-1}{m} > n^s - 1$. It follows that $n_1 > n^s$, so that at least one congestion-averse player uses resource 1. Since, by the equilibrium condition, that player does not want to move to one of the other resources, which have fewer users, it must be that $n_2 = \cdots = n_m = n_1 - 1$. However, this means that the total number of players satisfies (ii), a contradiction. This contradiction proves that an equilibrium actually does not exist. ■

When an equilibrium does exist, it is unique up to permutations of resources and of players of the same type. This assertion is a corollary of the following three observations, which pertain to any equilibrium and follow immediately from the definition. See Figure 1.

Observation 1. All congestion-seeking players use the same resource, which is the unique resource j_{max} with a maximal number of players:

$$n_{j_{max}} \geq n_j + 1, \qquad j \neq j_{max}\,.$$

Observation 2. For any two resources j and j' such that at least one congestion-averse player uses j,

$$n_j - n_{j'} \leq 1.$$

Observation 3. As a corollary of Observations 1 and 2, the inequality

$$n_{j_{max}} > n^s$$

holds (that is, some congestion-averse player uses j_{max}) if and only if $n^s \leq \frac{n-1}{m}$. In this case, each of the resources other than j_{max} has precisely $n_{j_{max}} - 1$ (congestion-averse) users.

One may wonder whether the conditions in Theorem 1 actually imply a stronger property than the existence of equilibrium, namely, existence of a strong or (at least) coalition-proof equilibrium. In a *strong equilibrium* (Aumann, 1959), beneficial deviations do not exist, not only for individual players but also for coalitions. *Coalition-proof equilibrium* (Bernheim et. al., 1987) is a weaker solution concept in which beneficial coalitional deviations may exist but they are not self-enforcing.

Holzman and Law-Yone (1996) study the conditions for the existence of strong equilibrium in congestion games with congestion-averse players (i.e., decreasing utility functions) in which strategies are sets of resources. They observe that a strong equilibrium always exists in the singleton case (where, as in our model, all strategies are singletons). They moreover prove that, in this case, every Nash equilibrium is strong. Rosenfeld and Tennenholtz (2006) consider the case of congestion-seeking players (increasing utility functions) and show that, essentially, the only strategy spaces that guarantee existence of strong equilibrium are those with singleton strategies. The necessity of the singleton condition is due to the extremely strong sense of "guaranteeing" in their model. For a given collection of strategies, they consider all ways of deciding which strategies are allowed for each player, and require that all corresponding congestion games possess strong equilibria. In summary, in the singleton-strategies case of both models, the set of strong equilibria is nonempty. Konishi et al. (1997) prove that, moreover, this set coincides with that of all coalition-proof equilibria.

The next proposition shows that, formally, the same coincidence also holds in our model, which differs in considering singleton congestion games with both congestion-averse and congestion-seeking players. However, it holds largely vacuously. With the exception of the special case of a single congestion-seeking player, strong and coalition-proof equilibria actually do not exist in our games.

Proposition 1. Consider a game G with congestion-averse and congestion-seeking players.

1. If $n^s = 1$, then every pure-strategy Nash equilibrium is strong.
2. If $n^s > 1$, then the game does not have a strong, or even coalition-proof, equilibrium.

In the following, we use the notation n_j^a and n_j^s for the number of congestion-averse and congestion-seeking users, respectively, of a resource j.

Proof. 1. Suppose that $n^s = 1$, and consider any Nash equilibrium. We have to show that, in the equilibrium, a coalitional deviation that is profitable to every coalition member does not exist. Suppose the contrary, that such a deviation does exist. As we show below, this assumption leads to a contradiction whether the coalition is *heterogeneous*, that is, consisting of both congestion-averse and congestion-seeking players, or *homogeneous*, and includes congestion-averse players only.

Suppose that the deviating coalition is heterogeneous. By Observation 1, its single congestion-seeking member deviates from the unique resource j_{max} with a maximal number of players to another resource j'. Since the deviation is profitable,

$$\tilde{n}_{j'} > n_{j_{max}}\,, \tag{1}$$

where n_j and $\tilde{n}_j$ are the number of players using a resource j before and after the coalitional deviation, respectively. Since $n^s = 1$, it follows from (1) and the definition of j_{max} that at least one congestion-averse player also moved to j', from another resource j''. Since this deviation is also profitable, necessarily

$$\tilde{n}_{j'} < n_{j''}. \tag{2}$$

However, inequalities (1) and (2) together contradict the maximality of $n_{j_{max}}$.

Suppose now that the deviating coalition consists of congestion-averse players only. The assumption implies that the coalitional deviation did not increase the number of players using any resource. This is because a congestion-averse player who moved to a resource where the number of users increased could have achieved at least as much by moving alone to that resource, contradicting the equilibrium assumption. Since the total number of players did no change, the number of users of each resource did not decrease either. Therefore, the coalitional deviation must be a permutation of its members' choice of resources, which implies that their total payoff did not change. However, this conclusion contradicts the assumption the deviation benefited them all. A similar argument is used in the proof of Holzman and Law-Yone's (1996) Theorem 2.1.

2. Suppose now that $n^s > 1$, and consider any equilibrium. To show that the equilibrium is not coalition-proof, it suffices to prove the same for the congestion-seeking players' strategies in the (n^s-player) subgame defined by fixing the strategies of the congestion-averse players. The strategies of the congestion-seeking players do not constitute a coalition-proof equilibrium because, as we show below, they can increase their payoffs to the maximum possible payoff in the subgame by deviating together from their common resource j_{max} (see Observation 1) to a resource j' with a maximal number of congestion-averse players:

$$n^a_{j'} = \max_j n^a_j.$$

To prove this assertion, it suffices to show that

$$n^a_{j'} > n^a_{j_{max}}. \tag{3}$$

If $n^a_{j_{max}} = 0$, inequality (3) holds trivially, since $n^a \geq 1$. Assume then that $n^a_{j_{max}} > 0$. By Observation 2, $n_{j_{max}} - n^a_{j'} \leq 1$. Rearranging and using $n_{j_{max}} = n^a_{j_{max}} + n^s$ gives

$$n^a_{j'} - n^a_{j_{max}} \geq n^s - 1.$$

Considering the assumption $n^s > 1$, this inequality proves (3). ■

4. CONVERGENCE TO EQUILIBRIUM

This section studies convergence to equilibrium under the assumption that players change their choice of resources one after the other. A finite sequence of strategy profiles obtained by such unilateral deviations is called a *path*. If the first and last strategy profiles are identical, the path is called a *cycle*. An *improvement* path or cycle is defined by the minimal rationality requirement that each deviation along the path is an improvement: it increases the deviating player's payoff. A *best-(response) improvement* path or cycle is one that satisfies the additional requirement that each deviating player's move is a best-improvement move: his choice of resource is a best response to the other players' choices. Non-existence of improvement cycles is a stronger property than non-existence of best-improvement ones. In a finite game, the first property is equivalent to the finite improvement property (see Section 1). The second property is referred to in this paper simply as *acyclicity*. Thus, a finite game is *acyclic* (with respect to best-improvement moves) if it does not have any best-improvement cycle. (Note that this definition is different from that of Young, 1993. The same applies to the definition of weak acyclicity below.)

A game with congestion-averse and congestion-seeking players with more than two resources does not have the finite improvement property. Indeed, such a game always has an improvement cycle similar to the one in the following example.

Example 1. Suppose that $n^a = 1, n^s = 3$ and $m = 3$. The congestion-averse player uses resource 1, one congestion-seeking player uses resource 2, and the other two use resource 3. The following is a better-response cycle: The first congestion-seeking player moves to resource 1, the congestion-averse player moves to resource 2, the congestion-seeking player returns there, and the congestion-seeking player returns to resource 1, thus completing the cycle.

In Example 1, the congestion-seeking player never chooses his best-response strategy, which is moving to the resource with the other two congestion-seeking players. If he did so, the cycle would be broken and an equilibrium would be reached. This observation leads to the question of whether the existence of cycles persists under best improvements, which is addressed by the following theorem.

Theorem 2. A game with congestion-averse and congestion-seeking players is acyclic if and only if condition (ii) in Theorem 1 holds, that is, $n \equiv 1 \bmod m$.

An immediate corollary of the theorem is that, in the special case of only two resources, the game has the finite improvement property if and only if the number of players is odd.

Proof. ($\Longleftarrow$) Suppose that $n \equiv 1 \bmod m$, so that $k = \frac{n-1}{m}$ is an integer. For each strategy profile, let $K^- = \{ j \mid n_j \leq k \}$ be the set of resources with k or fewer users and $K^+ = \{ j \mid n_j > k \}$ the resources with more than k users. These sets cannot be empty. $K^- = \emptyset$ would mean that the number of players in the game is at least $(k+1) \cdot m = n - 1 + m$, which is higher than the actual number n since $m > 1$. Similarly, $K^+ = \emptyset$ would mean that the number of players is at most $k \cdot m = n - 1$.

Claim 1 In a best-improvement path, no congestion-averse player moves to a resource in K^+ and no congestion-seeking player moves to a resource in K^-.

This result in an immediate corollary of the non-emptiness of K^- and K^+.

Claim 2. In a best-improvement path, no resource shifts from the set K^- to K^+. In a best-improvement cycle, the sets K^+ and K^- moreover never change.

A resource j can shift from K^- to K^+only as the result of a move to j of some player i who becomes the $(k+1)$-th user of j. By claim 1, player i is necessarily congestion-averse. Since the deviation is a best-improvement move, there must be at least $k+1$ other players in the resource that i comes from and at least k players in every other resource. However, this means that the total number of players is at least $k \cdot m + 2$, which contradicts the fact that $n = k \cdot m + 1$. The contradiction proves that no resource shifts from K^- to K^+.

To complete the proof of Claim 2, it only remains to note that, in a best-improvement cycle, no resource j can shift also in the opposite direction, from K^+ to K^-. This is because, later in the cycle, j would have to return from K^- to K^+.

By a similar argument, Claims 1 and 2 imply the following.

Claim 3. In a best-improvement cycle, no congestion-averse or congestion-seeking player moves *from* a resource in K^+ or in K^-, respectively.

By the above claims, in any best-improvement cycle, congestion-averse and congestion-seeking players move only within K^- and K^+, respectively. As we show below, this means that the following expression must increase after each move:

$$P = \sum_{j \in K^+} n_j^2 - \sum_{j \in K^-} n_j^2.$$

Suppose that a congestion-averse player moves from a resource $j' \in K^-$ to a resource $j'' \in K^-$. Since the player's move is an improvement, the following inequality must holds before it is performed:

$$n_{j'} > n_{j''} + 1.$$

The inequality implies that

$$n_{j'}^2 + n_{j''}^2 > n_{j'}^2 + n_{j''}^2 - 2(n_{j'} - n_{j''} - 1) = (n_{j'} - 1)^2 + (n_{j''} + 1)^2, \tag{4}$$

which proves that the change in P is positive.

Similarly, a move of a congestion-seeking player from a resource $j' \in K^+$ to a resource $j'' \in K^+$ is an improvement only if, before the move,

$$n_{j'} < n_{j''} + 1,$$

which again implies that the corresponding change in P is positive. Thus, P increases after each player's move along the cycle. However, this conclusion contradicts the fact that P must have the same value at the beginning of the cycle and at its end. The contradiction proves that the game is acyclic.

($\Longrightarrow$) Suppose that $n \not\equiv 1 \bmod m$. We have to show that a best-improvement cycle exists. Distribute the players among resources as equally as possible, so that, for some integer k, either all resources have k users or some of them have k and the others $k + 1$ users. From the assumption $n \not\equiv 1 \bmod m$ it follows that there must be at least two resources, j' and j'', with a maximal number of players (either k or $k + 1$). Assuming, without loss a generality, that some congestion-seeking player i' uses j' and some congestion-averse player i'' uses j'' (see example in Figure 2(A)), the following is a best-improvement cycle: i' moves to j'', i'' moves to j', i' returns to j', and i''returns to j'', thus completing the cycle. ■

The cycle presented at the last part of the proof can be broken. For example, in Figure 2, if *all* the congestion-seeking players followed player i' and moved to resource j'' (Figure 2(B)), and only then the congestion-averse players moved, an equilibrium would necessarily be reached (Figure 2(D)). Thus, the initial strategy profile in this example is connected to an equilibrium by *some* best-improvement path. The following theorem shows that this observation can be generalized. A game is said to be *weakly acyclic* if, starting at any strategy profile, there is some best-improvement path that ends in an equilibrium.

Theorem 3. Every game with congestion-averse and congestion-seeking players that has an equilibrium is weakly acyclic. Moreover, in such a game, every strategy profile is the starting point of some best-improvement path that ends in an equilibrium and in which each player moves at most once.

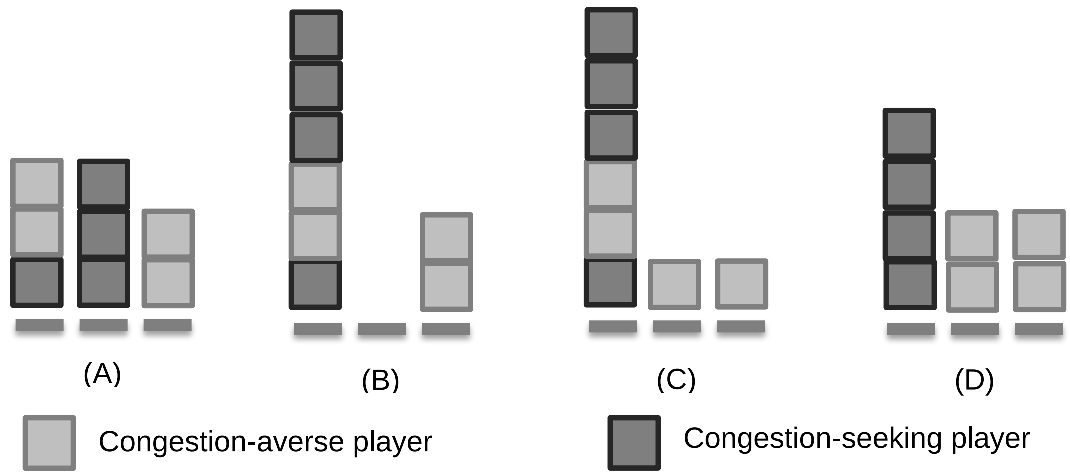

Figure 2. A best-improvement path in a game with $n^a = 4$, $n^s = 4$ and $m = 3$. Beginning with the initial state (A), three congestion-seeking players move one after the other and join the fourth one in the left-most resource, which becomes the one with the largest number of players (B). Their place is then taken by one of the congestion-averse players not using that resource (C). Finally, the two congestion-averse players in the left-most resource leave it and move to other resources. The outcome (D) is an equilibrium.

Proof. We define a *prioritizing algorithm* (or a *scheduler*; see Apt and Simon, 2012) and show how it helps players to avoid cycles and reach an equilibrium. At each step, the algorithm assigns a priority, a (the highest), b or c, to each player according to the following criteria:

a) Congestion-seeking players.
b) Congestion-averse players not sharing a resource with congestion-seeking ones.
c) Congestion-averse players sharing a resource with at least one congestion-seeking player.

One player then makes a best-improvement move. His identity is constrained only by the condition that no higher-priority player wants to move (that is, has a best-improvement move). The algorithm stops when no player wants to move, which means that an equilibrium was reached. See example in Figure 2.

To prove that the algorithm *necessarily* stops, it suffices to present a function P over the set of strategy profiles that increases at each step in the process. Such a function is defined by

$$P = \bar{n}^s_{j_{max}} + \frac{1}{\sum_{j=1}^{m} n_j^2},$$

where $\bar{n}^s_{j_{max}}$ is the number of congestion-seeking players using the resource with the largest number of users, and if there are several resources with a maximal number of users, the one with the smallest number of congestion-seeking users.

There are three cases to consider, a, b and c, according to the priority of the moving player.

a) A congestion-seeking player moves to a resource j_{max}, which necessarily has a maximal number of users and, by definition, at least $\bar{n}^s_{j_{max}}$ congestion-seeking ones. After the move, resource j_{max} becomes the *unique* resource with a maximal number of players,

$$n_{j_{max}} > \max_{j \neq j_{max}} n_j \ , \tag{5}$$

which implies that $\bar{n}^s_{j_{max}}$ has increased by at least 1. Since the second term in P is always positive and less than 1, the total change in P must be positive.

b) Since, by assumption, no congestion-seeking player wants to move, all of them are using the unique resource j_{max} with a maximal number of players. Therefore, the first term in P is equal to n^s. A congestion-averse player i moves from a resource $j' \neq j_{max}$ to some resource j'' with $n_{j'} > n_{j''} + 1$. After the move, inequality (5) still holds, which implies that the first term in P did not change. The change in the second term is given by the following expression, which by (4) is positive:

$$\frac{1}{\left(n_{j'} - 1\right)^2 + \left(n_{j''} + 1\right)^2 + \sum_{j \neq j', j''}^{m} n_j^2} - \frac{1}{\sum_{j=1}^{m} n_j^2} .$$

c) A congestion-averse player i moves from the resource j_{max} used by all the congestion-seeking players to some other resource, with has less than $n_{j_{max}} - 1$ users. Before the move, inequality (5) holds. As we show below, the same is true after i's move, which implies that the first term in P did not change. By the same argument used in case b, the second term increased, and therefore the same is true also for P itself.

By assumption, no congestion-averse player using a resource different from j_{max} wants to move, and since the move of (the congestion-averse) player i is a best-response one, the same is true after his move. Therefore, after the move,

$$\left| n_{j''} - n_{j'''} \right| \leq 1 \tag{6}$$

for all j'', $j''' \neq j_{max}$. Suppose that (5) does not hold after i's move, so that there are then $r > 1$ resources where the number of players is maximal, and equal to

$$q \stackrel{\text{def}}{=} n_{j_{max}} - 1 \geq n^s .$$

It then follows from (6) that the other $m - r$ resources have $q - 1$ users each. Summing up the numbers of users, we get that

$$n = (q - 1) \cdot m + r. \tag{7}$$

Since $r > 1$, it follows from (7) that condition (ii) in Theorem 1 does not hold. Condition (i) also does not hold, since

$$n^s \leq q = \frac{n-r}{m} + 1 < \frac{n-1}{m} + 1.$$

This contradicts the assumption that an equilibrium exists. The contradiction proves that (5) does in fact hold after player i's move, and completes the analysis of case c.

The analysis of the three possible cases proves that, if the players move according to the prioritizing algorithm, P always increases. As indicated, this implies that the algorithm necessarily stops, and ends in an equilibrium. We can now complete the proof of the theorem by showing that, moreover, each player moves at most once.

The first to move are the congestion-seeking players, who gather at a single resource j_{max}. After the last of their moves, inequality (5) holds. (The inequality holds also if the congestion-seeking players are at a single resource j_{max} already at the beginning and they do not want to move.) Then, the congestion-averse players move. As shown above, (5) still holds after each such move, and therefore the congestion-seeking players never have an incentive to move again. It remains to show that any congestion-averse player who moves also never wants to move again. The argument below is similar to that used by Fotakis (2010, Lemma 1).

Consider a congestion-averse player i that has just performed a best-improvement move to a resource j, and thus has no incentives at the current time to move again. We will show that player i will not have any incentives to move also at any later time. Suppose this is not so, and consider the first time resource j is not player i's optimal choice, but some other resource j' is optimal. This means that another congestion-averse player i' has either just moved to resource j or moved from j' to some third resource $j'' \neq j$. However, in the first case, player i' also would be better off choosing j' rather than j, a contradiction to the assumption that his choice of j was a best response. The second case contradicts the assumption that the move of player i' was an improvement, since it implies that player i would also be better off moving to j''. Indeed, the increase in the payoff of player i from doing so would be the sum of the increase in the payoff of player i' from moving from j' to j'' and the increase in his own payoff from his subsequent move from j to j'. These contradictions prove that player i will not in fact have an incentive to move again. ■

The proof of Theorem 3 presents a so-called *weak potential*: a real-valued function P over the set of strategy profiles with the property that, at every strategy profile that is not an equilibrium, *some* player has a best-improvement move that increases P. In a finite game, the existence of a weak potential implies weak acyclicity (since every best-improvement path along which P increases cannot visit the same strategy profile twice), and the converse implication holds as well (Kukushkin, 2004).

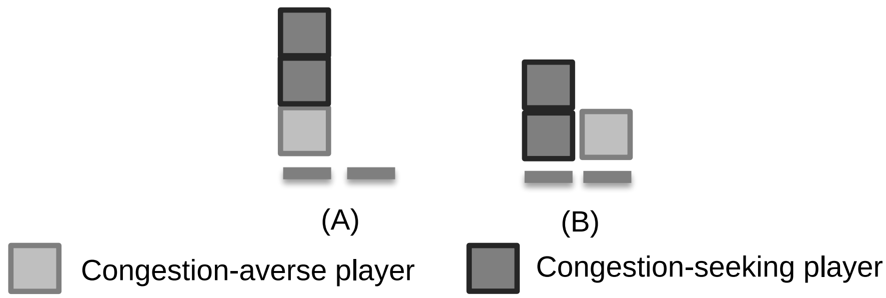

Figure 3. A subgame perfect equilibrium path in a sequential-move version of a game with one congestion-averse and two congestion-seeking players. The congestion-averse player enters first. Then a congestion-seeking player enters, and correctly predicts that if he chooses the same resource the first player chose, the third player will also do so. The result (A) is not an equilibrium in the simultaneous-move game, since a move of the congestion-averse player to the second resource would increase his payoff – and result in an equilibrium (B).

5. THE SEQUENTIAL MOVE GAME

The previous section is concerned with games where players choose resources *myopically*, that is, they maximize their payoff right after making the choice but do not consider the possible responses of the other players. In addition, each player may change resources multiple times. In this section, we explore the ability of players to reach an equilibrium by choosing resources sequentially and irrevocably: once a resource is selected, it cannot be changed. After all the players have chosen their resources, each player's payoff is determined according to his utility function. Crucially, we assume that players are forward looking, and strive to predict the choices of their followers. Specifically, we look for a subgame perfect equilibrium in a sequential-move version of the game (see Section 1). That game, and therefore also its set of subgame perfect equilibria (which in nonempty, as is the case for any finite perfect-information extensive form game), is determined by the players' entering order. Our main concern is with the equilibrium paths of these equilibria, that is, the players' actual choice of resources.

As the example in Figure 3 shows, the equilibrium path of a subgame perfect equilibrium is not necessarily an equilibrium in the simultaneous-move game. Moreover, for the sequential-move version considered in that example, *no* subgame perfect equilibrium gives an equilibrium in the simultaneous-move game. (Therefore, that game is not *sequentially solvable*; see Milchtaich, 1998.) However, such a subgame perfect equilibrium would exist if we changed the entering order by letting the two congestion-seeking players enter first. The equilibrium in Figure 3(B) would then be a subgame perfect equilibrium outcome. As we show below, an entering order whereby congestion-seeking players precede congestion-averse ones always guarantees the existence of *some* subgame perfect equilibrium whose equilibrium path is an equilibrium in the original game, if the set of equilibria in that game is nonempty. Moreover, *every* equilibrium in that set can be obtained this way.

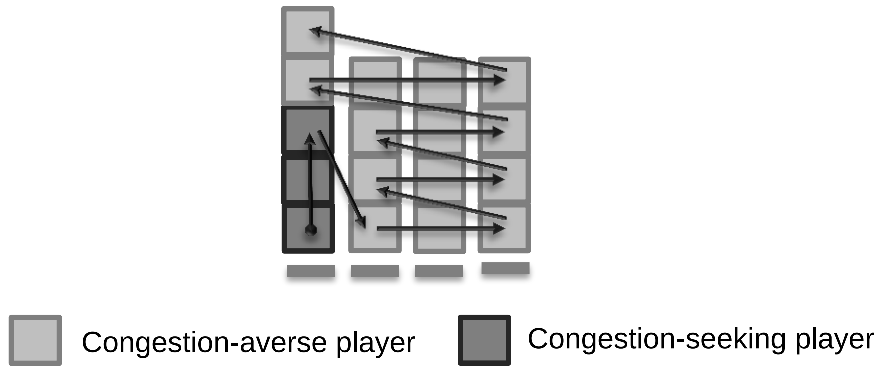

Figure 4. The depicted configuration is an equilibrium in the (simultaneous-move) game. It is also the equilibrium path of a subgame perfect equilibrium in the sequential-move version of the game where the players' entering order is that indicated by the arrows (so that, in particular, the first to choose their resources are the congestion-seeking players).

An equilibrium in a strategic game is said to be a *sequential-move equilibrium* (Milchtaich, 1998) if it coincides with the equilibrium path of some subgame perfect equilibrium in some sequential-move version of the (simultaneous-move) game.

Theorem 4. Every pure-strategy Nash equilibrium in a game with congestion-averse and congestion-seeking players is a sequential-move equilibrium.

The players' forward-looking behavior in the sequential-move game may seem to be at odds with the simple, myopic response of choosing an optimal resource given the choices of the preceding players (that is, a resource with a minimal or maximal number of users, depending on the entering player's type). However, the proof of the theorem shows that the two different manners of choosing resources may be reconciled. Specifically, the subgame perfect equilibrium strategies constructed in the proof always specify that an entering player reacts optimally to his predecessors' choices. Moreover, this is so also off-equilibrium, that is, after one or more of the previous players does not act according to his strategy. The players' strategies nevertheless incorporate an effective punishing mechanism, which guarantees that no single player can gain from choosing a different resource than that specified by his strategy.

Proof of Theorem 4. Let an equilibrium in the game be given. Without loss of generality, it may be assumed that the resources are indexed in such a way that $n_1 \geq n_2 \geq \cdots \geq n_m$. It may also be assumed that the players are indexed as follows. Players $1, 2, \dots, n^s$ are congestion-seeking (and therefore, by Observation 1, use resource 1). The congestion-averse players $n^s + 1, \dots, n^s + m - 1$ use resources $2, \dots, m$, respectively, as do players $n^s + m, \dots, n^s + 2m - 2$, and so on. This numbering scheme continues up to player $\min\{n, mn^s\}$. The remaining congestion-averse players, if any, are indexed in such a way that players $mn^s + 1, \dots, mn^s + m$ use resources $1, \dots, m$, respectively, and so on (see Figure 4).

In the sequential-move version of the game in which the players enter in the order $1, 2, \dots, n$, the following two rules recursively define a strategy for each player. As we show below, this strategy profile is a subgame perfect equilibrium whose equilibrium path coincides with the given equilibrium (in the simultaneous-move game).

BEST-RESPONSE RULE. Given the choice of resources by the preceding players, consider the set J of all resources yielding maximum payoff (which, for a congestion-averse or congestion-seeking player, are the resources with a minimal or maximal number of users, respectively). Choose the "left-most" resource in J, that is, the one with the smallest index.

PUNISHMENT RULE. This rule describes an exception to the previous one, which only applies if $|J| \geq 2$ and at least one of the preceding players who use a resource in J is a congestion-averse player who violated his strategy by choosing that resource. In this case, choose the same resource as the *last* such violator.

If the players follow these rules, then by Observations 1, 2 and 3 in Section 3 the result is the given equilibrium. It remains to show that the strategy profile specified by the rules is a subgame perfect equilibrium. Thus, it has to be shown that, regardless of the choices of the previous players, an entering player cannot benefit from violating the rules and choosing a resource j'' different from the resource j' prescribed by them, assuming that all later entrants will follow the rules.

For a resource j, set $n_j(0) = 0$ and, for $1 \leq l \leq n$, let $n_j(l)$ and $n_j^s(l)$ be the number of players and congestion-seeking players, respectively, using resource j right after player l enters the game, if player i chooses resource j'. Let $\tilde{n}_j(l)$ and $\tilde{n}_j^s\,(l)$ be defined similarly, accept that now player i chooses resource j''.

Assume first that player i is congestion-averse. To prove that i will not gain from choosing j' instead of j'', we have to show that the inequality

$$n_{j'}(l) \leq \tilde{n}_{j''}(l) \tag{8}$$

holds for $l = n$. For $l = i$, (8) holds by the Best-Response Rule. If this is not so for some larger l, then the largest $i \leq l \leq n-1$ for which (8) does hold satisfies

$$n_{j'}(l) = \tilde{n}_{j''}(l)$$

and

$$n_{j'}(l+1) = \tilde{n}_{j''}(l+1) + 1.$$

The two equalities means that (i) if player i chooses resource j', then player $l+1$ also chooses j', but (ii) if i chooses j'', then $l+1$ does not choose j''. It follows from (i) and the Best-Response Rule that $n_{j'}(l) \leq n_j(l)$ for all j. Therefore, the number of players among $i, i+1, \dots, l$ who chose each resource j is at least $[n_{j'}(l) - n_j(i-1)]^+$ (where $[x]^+ = \max\{x, 0\}$ denotes the positive part of a number x). Summing up over all resources, we obtain:

$$\sum_j \left[n_{j'}(l) - n_j(i-1)\right]^+ \leq l - i + 1. \qquad (9)$$

It follows from (ii) and the Best-Response and Punishment Rules that there is some resource j''' with $\tilde{n}_{j'''}(l) < \tilde{n}_{j''}(l)$. Again by the Best-Response Rule, each resource j chosen by one or more of the players $i, i+1, \dots, l$ satisfies $\tilde{n}_j(l) \leq \tilde{n}_{j'''}(l) + 1$ $(\leq \tilde{n}_{j''}(l))$. These inequalities give

$$\sum_j \left[\tilde{n}_{j''}(l) - \tilde{n}_j(i-1)\right]^+ > l - i + 1 \qquad (10)$$

(where the strict inequality reflects the strict inequality that refers to resource j'''). However, since $\tilde{n}_j(i-1) = n_j(i-1)$ for all $1 \leq j \leq m$ and $n_{j'}(l) = \tilde{n}_{j''}(l)$, inequalities (9) and (10) contradict one another. The contradiction proves that (8) in fact does hold for all $l \geq i$, and in particular for n. This proves that a congestion-averse player cannot gain from deviating from his strategy.

Suppose now that player i is congestion-seeking. If he follows his strategy and chooses resource j', it becomes the unique resource with a maximal number of users, so that every congestion-seeking player who enters after i also chooses j'. If i chooses j'', the following congestion-seeking players may still choose j' but they may also choose j''. Thus, whether player i chooses j' or j'', no resource ends up having more than $n^s_{j'}(n)$ congestion-seeking users. There are now two cases to consider, according to the size of this bound.

CASE 1: $n^s_{j'}(n) > n/m$.

To prove that, in this case, the congestion-seeking player i does not gain from choosing resource j'' instead of j' it suffices to show that

$$\max_j \tilde{n}_j(n) \leq n^s_{j'}(n).$$

If this inequality does not hold, then by the Best-Response Rule, applied to the congestion-averse players,

$$\min_j \tilde{n}_j(n) \geq n^s_{j'}(n).$$

However, this inequality contradicts the assumption that $n^s_{j'}(n) > n/m$.

CASE 2: $n^S_{j'}(n) \leq n/m$.

Whether player i chooses j' or j'', the assumed inequality and the Best-Response Rule imply that, right before or at some point after the congestion-averse players start entering the game, the following situation occurs: all the resources have the same number of users, and that number is $n^S_{j'}(n)$. Then, the remaining congestion-averse players, if any, enter the game according to the Best-Response and Punishment Rules. From the latter it follows that $\tilde{n}_{j''}(n) \leq n_{j'}(n)$. ■

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