Optimal management of fringe entry over time

Gila E. Fruchtera,∗;1 Paul R. Messingerb

aDepartment of Marketing, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
bSchool of Business, University of Alberta, Edmonton, Alberta, Canada

Abstract

In this paper, we investigate the problem of a dominant company facing entry of a “competitive fringe” (smaller competitor or fringe of smaller competitors). We seek to identify pricing and advertising (or other promotional strategies) that maximize long-term profits for the dominant firm, under possible reactions of the competitive fringe. Two main situations are considered:

• The firms in the fringe are price-takers, but they advertise.
• The firms in the fringe are not price-takers and advertise.

The possibility of a passive reaction, in the case of a very small fringe, is considered as a particular case.

We assume that the rate of change of fringe sales is dynamically related to the current sales, price and advertising efforts of both the dominant firm and the fringe. The higher the dominant firm price, the faster fringe entry. The higher dominant firm advertising effort, the slower fringe entry. Fringe advertising and pricing may counterbalance these effects. Formulating a dynamic game, with the dominant firm as a leader and the fringe as a follower, we present a new methodology for providing time-invariant feedback Stackelberg equilibrium. The methodology relies on finding the relationship between the co-state variables and the state variable. The equilibrium solution is obtained in an implicit form by solving a set of two backward differential equations. To show the applicability of our solution to real situations, we use data from the U.S. long-distance market and find optimal decision rules for AT&T facing the entry of MCI and Sprint during the 1980–1990 period. The feedback equilibrium indicates that while AT&T’s price is decreasing when fringe (MCI and Sprint) sales increase, the fringe price is increasing. AT&T’s advertising is increasing with fringe sales while the fringe’s advertising increases and then decreases. The comparison with actual behavior indicates that AT&T has adhered closer to the optimal solution in both price and advertising than the fringe.

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∗Corresponding author. Tel.: +852-235-7696; fax: +852-2358-2429.
E-mail address: fruchtg@mail.biu.ac.il (G.E. Fruchter).

1Present address: Graduate School of Business Administration, Bar-Ilan University, Ramat-Gan 52900, Israel.
1. Introduction

This paper deals with the marketing issue of how an entrenched firm, termed as a dominant firm, should respond when it is faced by potential entry into the product market. The situation of a dominant firm facing entry or expansion of a fringe of smaller competitors can arise in many circumstances—in nascent and newly decontrolled industries, when patents expire, when trade barriers fall, by historical accident, or when a dominant group of firms combine to maximize joint profit. The dynamics of such a situation are present whenever high prices of an industry leader induce competitive entry and the leader finds itself having to choose between short-run revenue and long-run market share. This issue would seem particularly pertinent in today’s context of the Internet and the World Wide Web.

A specific example of such a situation is the case of AT&T in the 1980s. For many years AT&T, being a government-regulated monopoly, was the dominant firm in the market for U.S. long-distance telephone service. In the 1970s and 1980s, however, court actions opened up the market by guaranteeing long-distance competitors access to local telephone exchanges and AT&T found itself facing a viable fringe of competitors. In 1981, AT&T enjoyed a market share of 97.01%, but two of its competitors, MCI and Sprint, were charging average long-distance rates that were about 29% lower. AT&T faced a dilemma: (1) if AT&T matched competitors’ prices, it would maintain market share, but lose 29% of revenue immediately; (2) if AT&T kept its prices high, it would maintain short-run revenue, but lose market share in the long-run. The question of what a dominant firm should do when facing such a dilemma is our primary focus of interest.

In this particular case, AT&T maintained and then gradually reduced prices, simultaneously increasing advertising. By 1990, AT&T rates were competitive with MCI and Sprint, and AT&T’s market share had fallen below 73% (see Table 1).

The question is whether AT&T behaved optimally. Motivated by this question, we introduce a methodology to manage such situations, and we particularly analyze this question in our empirical application.

Business history offers many other instances of dominant firms influencing competitive entry. For example, after a formative 1901 merger, U.S. Steel Corporation acted as price leader and, under its price umbrella, allowed gradual expansion of competitors in North America for several decades (Yamawaki, 1985; Parsons and Ray, 1975; Stigler, 1965). A more off-beat example consists of Reynolds International Pen Corporation, which sold its first ball-point pens (costing 80 cents to produce) at retail prices between $12 and $20 in 1945, but induced entry of some 100 competitors thereby, and watched its market share fall to zero by 1948 (Whiteside, 1951)! Dominant firms also influenced markets for mainframe computers (IBM in 1950–1970; Brock, 1975), copiers...
Table 1

U.S. long-distance market

<table>
<thead>
<tr>
<th>Year</th>
<th>Average AT&amp;T advertising pricea ($ million)b</th>
<th>Average fringe pricea ($ million)b</th>
<th>Fringe advertising (MCI &amp; Sprint) pricea ($ million)b</th>
<th>AT&amp;T market share (%)c</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>2.06</td>
<td>1.48</td>
<td>9.26</td>
<td>98.12</td>
</tr>
<tr>
<td>1981</td>
<td>2.17</td>
<td>1.57</td>
<td>14.88</td>
<td>97.01</td>
</tr>
<tr>
<td>1982</td>
<td>2.06</td>
<td>1.50</td>
<td>24.23</td>
<td>95.12</td>
</tr>
<tr>
<td>1983</td>
<td>1.99</td>
<td>1.53</td>
<td>40.06</td>
<td>92.36</td>
</tr>
<tr>
<td>1984</td>
<td>1.78</td>
<td>1.68</td>
<td>57.19</td>
<td>91.35</td>
</tr>
<tr>
<td>1985</td>
<td>1.62</td>
<td>1.55</td>
<td>58.29</td>
<td>91.00</td>
</tr>
<tr>
<td>1986</td>
<td>1.40</td>
<td>1.35</td>
<td>82.96</td>
<td>86.94</td>
</tr>
<tr>
<td>1987</td>
<td>1.15</td>
<td>1.12</td>
<td>88.36</td>
<td>84.01</td>
</tr>
<tr>
<td>1988</td>
<td>1.01</td>
<td>1.017</td>
<td>107.83</td>
<td>80.89</td>
</tr>
<tr>
<td>1989</td>
<td>0.90</td>
<td>0.86</td>
<td>123.96</td>
<td>76.37</td>
</tr>
<tr>
<td>1990</td>
<td>0.83</td>
<td>0.79</td>
<td>149.98</td>
<td>72.22</td>
</tr>
</tbody>
</table>

aBased on the average rates of three representative city pairs, across day, evening, and night rates, for calls of 5 and 10 minutes duration. Fringe price is the average of MCI and Sprint.
bAdjusted to 1980 dollars using the U.S. Consumer Price Index.
cAT&T quantity divided by quantity of AT&T, MCI, and Sprint.

(Xerox in the 1960s; Blackstone, 1972), rayon (American Viscose Company in 1919—after patents expired; Markham, 1952), “tin” cans (American Can Company in 1901; Hession, 1961), corn products refining, farm implements, synthetic fibers, aluminum extrusion, instant mashed potatoes, frozen orange juice, and automobiles (Scherer and Ross, 1990, Chapter 10). Indeed, the ubiquity of dominant firm situations suggests that further study of the dominant firm problem is worthwhile.

Modeling work related to the dominant firm problem dates back to Kamein and Schwartz (1971), who consider the case of probabilistic entry by a large competitor. They show that the incumbent firm’s optimal policy is to set the price at some appropriate constant level. A crucial assumption implicit in Kamein and Schwartz’s formulation is that entry occurs only in reaction to the price set by the incumbent. Bourguignon and Sethi (1981) relax this assumption by including advertising in their formulation. They show that under certain assumptions the optimal policy is to increase profit by charging a higher price and decrease the resulting threat of entry by a higher level of advertising.

The current paper develops a more general analysis of the dominant-firm problem. As with Bourguignon and Sethi, our model includes advertising as a control variable in addition to price; however, we consider a dynamic game formulation with possible strategic reaction of fringe firms.

Our work particularly builds on an influential paper by Gaskins (1971) wherein high prices of a dominant firm induce growth in sales of a “fringe” of smaller, non-strategic firms. We address a common criticism of Gaskins’ model, summarized by
Gilbert (1989) as follows:

A main criticism of Gaskins’ model is that entry is not an equilibrium process. The entry equation is not the result of optimizing decisions by a pool of potential entrants, but is specified exogenously in the model. The incumbent firm (or cartel) is presumed to act rationally, choosing a price policy to maximize present value profits. But there is no corresponding maximization problem for the firms that make up the flow of entry into the industry. Firms are not symmetric in the Gaskins model in their degree of rationality. Only the incumbent firm(s) can boast an identity in the Gaskins model. Entrants are relegated to a nameless component of an output flow.

We address this criticism by considering strategic profit-maximizing behavior of a market entrant, and how the dominant firm can optimally anticipate the entrant’s strategic reactions. As mentioned earlier, we include both advertising and price as strategic variables in our model, which also goes beyond Gaskins’ pricing analysis.

The current paper specifically develops a methodology to solve for a time-invariant feedback Stackelberg equilibrium of a dynamic game, with dominant firm as a leader and the fringe as a follower. The methodology relies on finding the relationship between the co-state variables and the state variable. The equilibrium solution is obtained in an implicit form by solving a set of two backward differential equations. The model and solution technique is applied in the empirical context of the U.S. long-distance telecommunications market in the 1980s. The feedback equilibrium indicates that while the dominant firm’s price is decreasing when fringe sales increase, the fringe price is increasing. The dominant firm’s advertising is increasing with the fringe sales while fringe’s advertising increases and then decreases. The comparison with actual behavior indicates AT&T has adhered closer to the optimal solution in both price and advertising than the fringe.

One of the earliest antecedents of the literature on dominant firms consists of the static dominant-firm/competitive-fringe model, first considered by Forchheimer (see Scherer and Ross, 1990, p. 224). The model assumes a “fringe supply curve” that describes fringe sales quantity as a function of the price of the dominant firm. The dominant firm’s sales quantity equals the residual of market demand and fringe supply. The dominant firm behaves strategically, as a monopolist would when facing this residual demand curve.

Our work also builds on the subsequent “limit-pricing” literature (Bain, 1956; Sylos-Labini, 1962; Modigliani, 1958). Unlike the static competitive-fringe model, these models recognize that high prices of the incumbent firm in the short run may induce entry into the market in the long run. The price that just forestalls entry is called the “limit price,” and this literature examines pricing at or near the limit price as a way of managing entry.

Lastly, our research is related to the literature on price leadership and advertising. Work in industrial organization distinguishes between dominant-firm, collusive and barometric price leadership (see Scherer and Ross, 1990, p. 248); the current paper describes a form of dominant-firm price leadership. The marketing literature provides fewer analyses of price leadership (an exception is Roy et al. (1994), which
describes collusive price leadership). Concerning advertising, while some authors focus on advertising as a signal (Milgrom and Roberts, 1986), our analysis is closer in spirit to work that connects advertising with limit pricing (Bagwell and Ramey, 1987). From another perspective, our analysis adds to the literature on dynamic advertising (Deal, 1979; Deal et al., 1979; Erickson, 1991, 1992; Chintagunta and Vilcassim, 1992; Fruchter and Kalish, 1997; Fruchter, 1999) by including dynamic pricing.

In the next section we set up the model.

2. The model: a Stackelberg game

We consider a dominant company that uses price and advertising (or other promotions) to cope with the entry of fringe firms into the market. For tractability, we neglect the effects of competitive actions within the fringe, itself, and treat the “competitive fringe” as a composite unit. The dominant company chooses its strategy, \((p^d, u^d)\), first, where \(p^d = p^d(t)\) is the price at time \(t\) and \(u^d = u^d(t)\) is advertising effort at time \(t\). The competitive fringe then reacts under one of the following three alternatives (see Fig. 1):

1. The firms in the “fringe” have no strategic role: they are price-takers and do not advertise. This would apply when the firms in the fringe are small and passive. In this case, fringe firms match the firm’s price strategy \(p^d\) and set advertising effort at zero. This implies a (simple) optimal control problem.

2. The firms in the “fringe” have a strategic role implying a dynamic hierarchical game-theoretic problem. That is, we consider a two-dimensional Stackelberg game, with the dominant company acting as the leader and the competitive fringe as the follower. We distinguish between two situations:

   A. Fringe firms are price-takers, but they advertise. Therefore, they react with a price strategy \(p^f = p^f(t)\) and an advertising effort \(u^f = u^f(t)\).

   B. Fringe firms are not price-takers and react strategically with a competitive pricing strategy, \(p^f = p^f(t)\), and an advertising effort \(u^f = u^f(t)\).

![Table](https://via.placeholder.com/150)

<table>
<thead>
<tr>
<th>ADVERTISING</th>
<th>YES</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>((p^d, u^f))</td>
<td>((p^d, 0))</td>
</tr>
<tr>
<td>NO</td>
<td>((p^f, u^f))</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1. Possible reactions of the competitive fringe.
We are interested in time-invariant feedback Stackelberg equilibrium, (see, e.g. Basar and Olsder, 1995). This means that the players design their strategic actions as state-dependent decision rules, \((p^d(x), u^d(x))\) and \((p^f(x), u^f(x))\). The state variable \(x = x(t)\) measures the current level (at time \(t\)) of fringe sales. Such strategies are desirable because they react to actual market conditions. The feedback Stackelberg equilibrium forms a set of optimal decision rules for dealing with entrants.

To assist the reader, Table 2 summarizes the nomenclature used in this paper.

We assume that the rate of change of fringe sales, \(\dot{x}(t) \equiv \frac{dx}{dt}\), is determined by current price and advertising of the dominant company and the current price and advertising response of the fringe, as well as the current level of fringe sales. In a general formulation (given a starting value of fringe supply of \(x_0\)), we assume

\[
\dot{x}(t) = g(x, p^d, p^f, u^d, u^f), \quad x(0) = x_0.
\]

Gaskins (1971) model assumes a passive reaction of the fringe and focuses only on the current price of the dominant company with the following specification for the rate of growth of fringe sales:

\[
\dot{x}(t) = k(p^d - \bar{p}). \tag{1a}
\]

His model assumes that the rate of entry is a monotonically nondecreasing (actually linear\(^2\)) function of current dominant company price; \(\bar{p}\) can be interpreted as the limit price below which exit from the market by fringe producers will occur.

\(^2\) Gaskins mentioned that his linear assumption can be viewed as a first-order approximation of a more complicated functional relationship between \(dx/dt\) and \( (p^d - \bar{p})\). In his unpublished Ph.D. dissertation he also considered a quadratic entry model.
Our model extends Gaskins’ model by developing a dynamic game that includes advertising. We assume that the function \( g \) is monotonically nondecreasing with respect to \( p^d \) and \( u^d \) and nonincreasing with respect to \( p^f \) and \( u^f \), i.e., \( \partial g / \partial p^d \geq 0 \), \( \partial g / \partial u^d \leq 0 \), \( \partial g / \partial p^f \leq 0 \) and \( \partial g / \partial u^f \geq 0 \). This means that higher dominant firm pricing induces greater fringe entry, but higher advertising effort can reverse this effect. Lower competitive fringe pricing and higher fringe advertising effort both lead to greater fringe sales.

Assuming that firms maximize the present value of their profit stream over an infinite horizon, the dominant firm maximizes \( \Pi_d \) with respect to \( p^d \) and \( u^d \), where

\[
\Pi_d = \int_0^\infty [(p^d - c_d)q - (u^d)^2]e^{-rt} \, dt, 
\]

\((u^d)^2\) represents the advertising expenditure, \(^3 c_d\) is the marginal production cost, and \( r \) is the common discount rate. In formula (2), \( q \) denotes the dominant company’s current sales; we further assume that \( q \) can be represented by the difference between the current total market demand, denoted by \( f \), and the current level of fringe sales, \( x \); i.e.,

\[
q = f(p^d, p^f, u^d, u^f) - x. 
\]

In formula (3), we assume that the total market demand, \( f \), depends on all price and advertising variables; in addition it is a twice differentiable function with respect to all variables, monotonically decreasing with respect to the price variables and nondecreasing with respect to advertising variables, i.e., \( \partial f / \partial p^d < 0 \), \( \partial f / \partial p^f < 0 \), \( \partial f / \partial u^d \geq 0 \), \( \partial f / \partial u^f \geq 0 \). In other words, marketing activities of both dominant firm and competitive fringe may expand the total market demand.

The competitive fringe maximizes \( \Pi_f \) with respect to \( p^f \) and \( u^f \), where

\[
\Pi_f = \int_0^\infty [(p^f - c_f)x - (u^f)^2]e^{-rt} \, dt, 
\]

\( c_f \) represents the marginal production cost, and \((u^f)^2\) represents the advertising expenditure.

Overall, the dominant company’s and competitive fringe’s optimization problem (1)–(4) can be summarized as the following differential game:

\[
\begin{align*}
\max_{p^d, u^d} \Pi_d &= \int_0^\infty [(p^d - c_d)f(p^d, p^f, u^d, u^f) - x - (u^d)^2]e^{-rt} \, dt \\
\max_{p^f, u^f} \Pi_f &= \int_0^\infty [(p^f - c_f)x - (u^f)^2]e^{-rt} \, dt
\end{align*}
\]

s.t. \( \dot{x} = g(x, p^d, p^f, u^d, u^f) \) and \( x(0) = x_0. \)

\(^3\) The assumption we make here that the advertising expenditure is the square of the advertising effort is commonly assumed in the advertising literature and incorporates the intuition of diminishing returns to advertising expenditures.
In the control theory framework (e.g., see Pontryagin et al., 1962; or Kamein and Schwartz, 1981), \( x \) is a state variable, \( p^d, p^f, u^d \) and \( u^f \) are control variables, \( x(0) = x_0 \) is the initial condition, and \( \dot{x} = g \) is the equation of motion.

### 3. Feedback Stackelberg equilibrium strategies

To find the feedback Stackelberg equilibrium of the differential game, we start by solving the follower’s problem, which is an optimal control problem. We compute the price and advertising effort of the competitive fringe as the rational reaction to the price and advertising effort used by the dominant company. Next, we solve an optimal control problem of the dominant company, taking into account the constraints on the dominant company and the reaction function of the competitive fringe. We now state the key relationships for the optimization problems; for more details see Appendix A.

Let \( \lambda_f \) be the co-state variable for the follower problem. Let also \( p^f^* \) and \( u^f^* \) be the optimal price and advertising solution for the follower. From the necessary and sufficient conditions of the follower’s optimization problem (see Appendix A), and using the implicit function theorem (see Apostol, 1979, p. 374) we are able to arrive at the unique follower (fringe) reaction strategy in terms of the state and co-state variables, \( x \) and \( \lambda_f \), and the dominant firm’s price and advertising, \( p^d \) and \( u^d \),

\[
p^f^* = p^f^*(x, \lambda_f, p^d, u^d) \quad \text{and} \quad u^f^* = u^f^*(x, \lambda_f, p^d, u^d). \tag{6}
\]

The equations

\[
\dot{x} = g(x, p^d, p^f^*(x, \lambda_f, p^d, u^d), u^d, u^f^*(x, \lambda_f, p^d, u^d)) = \tilde{g}(x, \lambda_f, p^d, u^d) \tag{8}
\]

and

\[
f(x, \lambda_f, p^d, u^d) = f(p^d, p^f^*(x, \lambda_f, p^d, u^d), u^d, u^f^*(x, \lambda_f, p^d, u^d)) \tag{9}
\]

then become constraints in the dominant firm’s optimization problem.

Now let \( \lambda_d \) be the co-state variable of the dominant firm (leader). From the necessary and sufficient conditions of the leader (dominant firm) optimization problem, see Appendix A, and using again the implicit function theorem, as above, we arrive at the unique leader strategy in terms of the state and co-state variables, \( x \) and \( \lambda_d \), and \( \lambda_f \),

\[
p^d^* = p^d^*(x, \lambda_d, \lambda_f) \quad \text{and} \quad u^d^* = u^d^*(x, \lambda_d, \lambda_f). \tag{10}
\]

In the following we will show that we can find two functions \( \Psi \) and \( \Phi \) such that the co-state variables will be functions of the state variable \( x \), i.e.,

\[
\lambda_d(t) \equiv \Psi(x(t)) \quad \text{and} \quad \lambda_f(t) \equiv \Phi(x(t)). \tag{10}
\]

In particular, \( p^d^* \), \( p^f^* \), \( u^d^* \), \( u^f^* \) are functions of the state variable, and they form a time-invariant feedback Stackelberg equilibrium strategy. This result is presented in the following theorem.

**The Main Theorem.** Consider the dominant firm’s and competitive fringe’s optimization problem. Using the notation described above, assume that the sufficient condition

### Appendix A

...
for the dominant firm holds in a neighborhood of \((p^d^*, u^d^*)\) and that the sufficient condition for the follower holds in a neighborhood of \((p^f^*, u^f^*)\). Let

\[
p^d^* = p^d^*(x, \Psi(x), \Phi(x)) \quad \text{and} \quad u^d^* = u^d^*(x, \Psi(x), \Phi(x)) \quad \text{and} \quad (11)
\]

\[
p^f^* = p^f^*(x, \Phi(x), p^d^*, u^d^*) \quad \text{and} \quad u^f^* = u^f^*(x, \Phi(x), p^d^*, u^d^*) \quad \text{and} \quad (12)
\]

where \(\Psi(x)\) and \(\Phi(x)\) satisfy the following system of backward differential equations

\[
\Phi'(x)g(x, \phi(x), p^d^*, u^d^*) = r\Phi(x) - c_d(1 - \tilde{f}_d(x, \Phi(x), p^d^*, u^d^*)) - \Psi(x)\tilde{g}_d(x, \phi(x), p^d^*, u^d^*) \quad \lim_{t \to \infty} \Psi(x(t))e^{-rt} = 0,
\]

\[
\Phi'(x)g(x, p^d^*, p^f^*, u^d^*, u^f^*)
\]

\[
= r\Phi(x) - (p^f^* - c_f) - \phi(x)[g_d(x, p^d^*, p^f^*, u^d^*, u^f^*)
\]

\[
+ g_{p^d^*}(x, p^d^*, p^f^*, u^d^*, u^f^*)(d p^d^*/dx) + g_{u^d^*}(x, p^d^*, p^f^*, u^d^*, u^f^*)(d u^d^*/dx)],
\]

\[
\lim_{t \to \infty} \Phi(x(t))e^{-rt} = 0 \quad (13)
\]

with \(\tilde{g}\) and \(\tilde{f}\) defined in (8) and (9). Then, \((p^d^*, u^d^*)\) constitutes a unique local time-invariant feedback Stackelberg strategy for the leader in the differential game described by (5), and \((p^f^*, u^f^*)\) is the unique optimal time-invariant feedback response of the follower.

**Proof.** See Appendix B. \(\square\)

In practice, this theorem is applied to obtain feedback solutions in four steps, as summarized in Fig. 2.

**Remark 1.** The strategy \((p^d^*, u^d^*, p^f^*, u^f^*)\) is the feedback Stackelberg equilibrium of the differential game (5). Note that it also forms a Nash equilibrium solution of (5), cf. Basar and Olsder (1995).

### 3.1. Some remarks on the methodology

The idea of generating a relationship between the co-state and state, as in (10), originated in the sweep method, see Gelfand and Fomin (1963, Chapter 6). An application of the sweep method for a linear-quadratic problem can be found in Bryson and Ho (1975). They use the method to solve a two-point boundary problem. By finding a relationship between the co-state and state they can find the values of the co-state at the initial time and, as a result, transform the two-point value problem into a initial-value problem that is easier to solve. The purpose in our paper is to go one step further. Following Fig. 2 we use the relationships between the co-states and state, \(\lambda_d = \Psi(x)\) and \(\lambda_f = \Phi(x)\) to find the decision control rules for the variables, \(p^d^*, u^d^*, p^f^*, u^f^*\), as functions of only the state variable, \(x\) (i.e., time-invariant state feedback controls).
STEP 1: Solve for \( p^f = p^f(x, \lambda_d, p^d, u^f) \) and \( u^f = u^f(x, \lambda_d, p^d, u^f) \), using the necessary optimality conditions for the fringe, and for
\[
\lambda_d = \lambda_d(x, \lambda_d, \lambda_f) \quad \text{and} \quad u^d = u^d(x, \lambda_d, \lambda_f),
\]
using the necessary optimality conditions for the dominant firm. (For necessary optimality conditions, see Appendix 1.)

STEP 2: Substitute \((p^f, u^f)\) and \((p^d, u^d)\) into
\[
g(x, p^d, p^f, u^d, u^f) = 0,
\]
\[
\lambda_d (p^d - c_d) (1 - \bar{f}^d(p^f, u^f)) - \lambda_f (x, p^f, p^d, u^d, u^f) = 0
\]
and
\[
\lambda_f (p^f - c_f) - \lambda_d (g_f + g_d dp^f/du^f + g_u du^d/du^f) = 0,
\]
and solve for \(x, \lambda_d\) and \(\lambda_f\).
(These equations arise after imposing the steady state conditions, \(\dot{x}(\infty) = 0, \dot{\lambda}_d(\infty) = 0\) and \(\dot{\lambda}_f(\infty) = 0\). See Appendix 1.)

STEP 3: Solve for the terminal values \(x(\infty), \lambda_d(\infty)\) and \(\lambda_f(\infty)\).

STEP 4: Solve the system (13) backward from \(\bar{V}_d(x) = \bar{\Psi}(\infty) = \lambda^d(\infty)\) and \(\bar{V}_f(x) = \bar{\Phi}(\infty) = \lambda^f(\infty)\). This yields a solution \((\bar{\Psi}(x), \bar{\Phi}(x))\). The feedback solutions \(p^d(x), p^f(x)\) and \(u^d(x), u^f(x)\) are obtained by plugging (successive values of) \(x, \lambda_d = \bar{\Psi}(x)\) and \(\lambda_f = \bar{\Phi}(x)\) into
\[
p^d = p^d(x, \lambda_d, \lambda_f), \quad u^d = u^d(x, \lambda_d, \lambda_f), \quad p^f = p^f(x, \lambda_d, \lambda_f, u^f) \quad \text{and} \quad u^f = u^f(x, \lambda_f, p^f, u^f).
\]

Fig. 2. How to apply the main theorem to find feedback solutions: the 4-step approach (to simplify the notation we remove the superscript * for the optimal solutions).

Two relationships \(\lambda_d = \Psi(x)\) and \(\lambda_f = \Phi(x)\) are found by solving the two backward differential equations in (13).

Dynamic programming interpretation. System (13) forms a set of necessary condition for the optimal feedback controls \(p^d(x), u^d(x), p^f(x), u^f(x)\). To see this let \(V^d(x)e^{-rt}\) and \(V^f(x)e^{-rt}\) be the value functions of the leader and follower respectively,\(^4\) and let us assume that \(V^d(x) = \Psi(x)\) and \(V^f(x) = \Phi(x)\). Then the Hamilton–Jacobi–Bellman (HJB) equation for the follower will be
\[
rV^f(x) = \Phi(x) g(x, p^d, p^f, u^d, u^f) + (p^f - c_f)x - (u^f)^2
\]
and for the leader will be
\[
rV^d(x) = \Theta(x) \bar{g}(x, \Phi(x), p^d, u^d) + (p^d - c_d)(\bar{f}(x, \Phi(x), p^d, u^d) - x) - (u^d)^2.
\]
Taking the derivative with respect to \(x\) of the last two equations and considering the optimality conditions defined in Appendix A by (A.2) and (A.3) for the follower

\(^4\)This form of value function arises from the fact that in the problem statement, the time variable \(t\) does not appear explicitly, except in the factor \(e^{-rt}\).
problem, and by (A.12) and (A.13) for the leader, respectively, one can arrive exactly at system (13).

Thus, the system in (13) forms a set of necessary conditions for the optimal feedback controls \( p^d^*(x), u^d^*(x), p^f^*(x), u^f^*(x) \) and we have arrived at the same conclusion that the system in (13) leads to a time-invariant feedback Stackelberg solution. In other words, we conclude that our method and dynamic programming lead to the same results.

In sum, the Main Theorem presents a methodology for finding time-invariant feedback solutions in an implicit form by solving a set of two backward differential equations.

Next we analyze two more possible fringe reactions.

3.2. The fringe is price-taker and advertises

In this case, the problem of the dominant firm is as above. However, we have a simpler problem to solve for the follower problem. Specifically, the necessary optimality conditions include only one equation, and similarly the sufficient condition, see Appendix A. Relations (6) and (7) are reduced to (7), and instead of (6) we have \( p^f^* = p^d^* \).

3.3. The fringe is passive: price-taker and does not advertise

In this case we only have the optimization problem of the dominant firm, i.e.,

\[
\begin{align*}
\text{Max } & \Pi_d = \int_0^\infty \left[ (p^d - c_d)(f(p^d, u^d) - x) - (u^d)^2 \right] e^{-rt} \, dt \\
\text{s.t. } & \dot{x} = g(x, p^d, u^d) \quad \text{and} \quad x(0) = x_0. 
\end{align*}
\]

(14)

For necessary and sufficient optimality conditions for this problem see Appendix A. The Main Theorem is reduced to the following theorem.

**Theorem 1.** Consider the profit-maximizing problem (14). Using the notation above, suppose the sufficient condition holds in some neighborhood of \((p^d^*, u^d^*)\). Then the pair \( p^d^* = p^d^*(x, \Psi(x)) \) and \( u^d^* = u^d^*(x, \Psi(x)) \), where \( \Psi(x) \) satisfies the following backward differential equation,

\[
\Psi'(x)g(x, p^d^*, u^d^*) = r\Psi(x) + p^d^* - c_d - \Psi(x)g_d(x, p^d^*, u^d^*),
\]

\[
\lim_{t \to \infty} \Psi(x)e^{-rt} = 0,
\]

forms a unique local optimal time-invariant feedback strategy.

**Proof.** Analogously to the Main Theorem. □
The equilibrium \((p^d^*, u^d^*)\) being the optimal strategy of the dominant company under possible reactions of the competitive fringe, \((p^f^*, u^f^*)\), forms a set of optimal decision rules for a dominant company dealing with entrants.

In Section 4 we apply our approach to a real-world example.

4. An empirical application to the U.S. telecommunications market

We now apply our theoretical results in a particular empirical context.

4.1. The data set

We use data descriptive of the U.S. telecommunications market from 1980 to 1990. AT&T, as a government-regulated monopoly, was for many years the dominant firm, but it found itself facing entry of competitors in the 1970s and 1980s after court actions opened up the U.S. long-distance telephone market to competition (by guaranteeing long-distance competitors access to local telephone exchanges).

We constructed a price index for long-distance telecommunication services using rates of three representative city pairs, across day, evening, and night rates for calls of 5 and 10 minutes duration. Thus, the index of long-distance prices for each company was the average of 18 rates for that company during the given year. The three representative city pairs that we used were Philadelphia–Detroit, Philadelphia–San Francisco, and Philadelphia–Dallas/Houston (the rate is the same to Dallas and to Houston). This selection of three city pairs was made from the 38 city pairs for which data are reported by the Federal Communications Commission in “Statistics of Communications Common Carriers” (1995/1996 Edition). These three city pairs were selected to connect an eastern city with a Midwestern city, a western city, and a southern city. The data begin in 1980 because that is the first year for which the latter publication includes pricing data for MCI and Sprint. The published prices were for December 31 for each associated year. The average of the year-end price of the previous year and the current year is calculated and used as the average price during the year \(t\). We then deflated these prices using the consumer price index (so that the resulting price index is in 1980 dollars). We denote the derived price index in year \(t\) for AT&T as \(p_t\). We did similar calculations for Sprint and MCI.

We operationalize fringe quantity \(x_t\), AT&T quantity \(q_t\), and market quantity \(f_t\), at year \(t\), as follows. We obtained annual revenues of MCI, Sprint, and AT&T from annual issues of Statistics of Communications Common Carriers, published by the U.S. Federal Communications Commission (Table 1.4, “Total Toll Services Revenues”). We also deflated these revenues using the consumer price index. We divided the deflated revenue for each company by the price index for that company (as described in the previous paragraph) to arrive at a quantity index for each company. Fringe quantity \(x_t\), is the sum of the quantity indices for Sprint and MCI. AT&T’s quantity, \(q_t\), is the quantity index for AT&T. The market quantity is taken as the sum of AT&T and fringe quantity, \(f_t = q_t + x_t\).
AT&T advertising expenditure at year $t$, $(u_d^t)^2$, was obtained in selected annual issues of *Advertising Age*, in their story listing the “100 largest advertisers” in the U.S., and deflated using the consumer price index (so that $(u_d^t)^2$ is in million US 1980 dollars). AT&T advertising effort, $u_d^t$, is taken as the square root of the advertising expenditure at year $t$. We obtained similar estimates of MCI and Sprint advertising effort $u_f^t$ as the square root of sum of the advertising expenditures of MCI and Sprint, deflated to 1980 dollars using the consumer price index.$^5$

4.2. Model specification: estimation of structural equations

The structural equations of our model consist of the dynamic equation (1) and the demand function equation (3). We focus on the particular formulation of the dynamic equation:

$$\left(\frac{\dot{x}}{x}\right)_t = k[(p_d^t)^2 - (p_f^t)^2] - c[u_d^t x_t - u_f^t (f_t - x_t)], \quad k > 0, \ c > 0.$$  \hspace{1cm} (16)

This model extends Gaskins’ model by being a price and advertising response model and by incorporating the competitive effects of the fringe. We concluded with this model after a long history of attempting to relate the data and the relevant variables. The first two terms extend Gaskins price response model by incorporating competitive pricing effects; also we use a quadratic form. The last two terms form a variation of the Lanchester advertising combat model (Kimball, 1957) using sales levels, as in Vidale and Wolfe (1957), instead of market share levels. The use of percentage changes seems plausible, since changes would occur proportional to the current state and not just in absolute terms (regardless of the current state). The discrete-time analog of (16) takes the form

$$\frac{x_{t+1} - x_t}{x_t} = k[(p_d^t)^2 - (p_f^t)^2] - c[u_d^t x_t - u_f^t (f_t - x_t)] + \varepsilon_{1t}.$$  \hspace{1cm} (17)

$^5$ Before 1990, MCI was not in the list of 100 top advertisers in the U.S. So we estimated 1982–1989 advertising as a proportion (based on the 1990 year) of the Sales, General and Administrative entry in the annual income statements in the annual reports; 1980–1981 advertising was based on a proportion of the sales and marketing entry in the income statements in the annual reports.

Before 1988, Sprint was not in the top 100 advertisers in the U.S. In addition, Sprint went through restructuring and mergers in the 1980s. In particular, GTE acquired smaller companies and formed GTE Sprint in the early 1980s. In 1986 and 1987, U.S. Sprint was formed as a merger of GTE Sprint and the long distance business of United Telecommunications (GTE and UT each owned 50%). In 1989, UT acquired a total of 80.1% of U.S. Sprint. (The 1988 annual reports, published in mid-1989 show Sprint as merged with UT.) In 1991, UT purchased from GTE the remaining 19.9%, thus completing the acquisition. In 1991, United Telecommunications changed its name to Sprint Corporation. These frequent changes make it difficult to use the accounting reports for consistent measures of advertising, although they do provide consistent measures of long-distance revenues. We estimated 1980–1987 Sprint’s advertising as a multiple of MCI’s ad/sales ratio (based on the 1988 year) times Sprint’s sales in 1980–1987.
Table 3
Parameter estimates

<table>
<thead>
<tr>
<th>Eq. (17)</th>
<th>Eq. (18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>Estimate</td>
</tr>
<tr>
<td>$k$</td>
<td>0.26</td>
</tr>
<tr>
<td>$c$</td>
<td>2.56E - 6</td>
</tr>
<tr>
<td>$c_d$</td>
<td>1.6</td>
</tr>
<tr>
<td>$c_f$</td>
<td>0.1</td>
</tr>
<tr>
<td>$x_0$</td>
<td>242.61</td>
</tr>
<tr>
<td>Outside estimates</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.34</td>
</tr>
</tbody>
</table>

*aThe units for the advertising variable for this estimation were millions of U.S. dollars (inflation adjusted to 1980) (as in Table 1). The units of market demand, $f$, and fringe sales, $x$, are in millions of quantity index units, which is descriptive of the quantity of long-distance calls in the U.S. The parameter values of Table 2 are reported on this basis.

*$x_0 = x(1980) = 242.6058$ million calls per year, from our quantity index.

We use a linear formulation for the demand function

$$f_t = a - b p_t^d + d_1 u_t^d + d_2 u_t^f + \varepsilon_{2t}.$$  \hfill (18)

As in Gaskins’ (1971) model, the dominant firm price determines market demand and the dominant firm price premium determines the rate of growth of the fringe sales. The error terms ($\varepsilon_{1t}, \varepsilon_{2t}$) are assumed to be distributed normally with mean (0, 0) and with the $(2 \times 2)$ covariance matrix $\Omega$.

Note that there may be contemporaneous correlation in the error terms $\varepsilon_{1t}$ and $\varepsilon_{2t}$, which implies that estimating (17) and (18) as a system leads to greater efficiency in estimation. Also note that, according to our model, the variables on the right-hand side of (17) and (18) are generated by simultaneous optimization of the two control problems in (5). A current realization of either error term (if observed) could, thus, affect the current optimal choices of the variables $p_t^d, p_t^f, u_t^d, u_t^f$ in Eqs. (17) and (18), engendering a possible simultaneity problem.

To improve the efficiency of estimation and to avoid simultaneity bias, we use three-stage least squares. We use one-period-ahead lead values of the right-hand side variables as instruments. The use of instrumental variables addresses the simultaneity problem. Often lagged right-hand side variables are used as instruments, but we do not have complete observations prior to 1980, so we find it more convenient to use lead variables. We used the SAS/ETS software (Proc Model) for the estimation.

Estimates of the parameters of system (17) and (18) are reported in Table 3. In a previous run, we found that we could not reject the hypotheses that $d_1 = d_2 = 0$. The $t$-statistics are significant and the $R$-squared statistic for Eq. (18) is good.
The low $R$-squared statistic for Eq. (17) indicates that the model variables may not be the only things affecting the rate of growth of the fringe; other variables possibly affecting the rate of growth of fringe sales include government telecommunications regulation, word-of-mouth diffusion effects, and increased usage of information technology relating to long distance telecommunications.

For completeness, Table 3 also includes outside estimates (managerial judgements) of the remaining parameters of the model, $c_d,c_f,r,$ and $x_0$. We will use the parameters from Table 3 to apply our methodology for arriving at a solution.

4.3. Application of the feedback Stackelberg equilibrium

We now apply the approach of Section 3 to derive the feedback equilibrium strategies, i.e. strategies that are functions of actual fringe sales, $x$. In the case of discrete time, this implies that the equilibrium price and advertising at any period of time are functions of fringe sales for that period.

We proceeded with the four steps of our methodology described in Fig. 2 using the software Mathematica and the parameter values in Table 3. The feedback equilibrium strategies for AT&T ($p^d(x),[u^d(x)]^2$) and the fringe ($p^f(x),[u^f(x)]^2$) are described in Fig. 3.
Table 4
Root relative mean squared error (RRMSE)

<table>
<thead>
<tr>
<th>Variable</th>
<th>AT&amp;T (the dominant firm)</th>
<th>MCI &amp; Sprint (the fringe firms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.86</td>
<td>0.89</td>
</tr>
<tr>
<td>Advertising</td>
<td>0.74</td>
<td>52.96</td>
</tr>
</tbody>
</table>

From Fig. 3 the optimal strategy for the leader (the dominant firm) and follower (fringe’s firms) can be characterized and interpreted as follows.

1. Dominant-firm price is decreasing. This represents a trade-off between short-run revenue and future market share. When price is high and fringe market share is low, then deterring further entry can be very costly because of the substantial price cuts and immediate loss in current revenues required in doing so. Maintaining high price, on the other hand, might allow the fringe to grow, but allowing fringe market growth is not nearly so costly in the short run. The dominant firm, thus, finds it advantageous to take short-run revenues and charge a high price. Later, once the fringe has gained share, allowing continued rapid growth in share is more costly to the dominant firm, and the dominant firm lowers price.

2. Fringe price is increasing. The incentives are the opposite for the fringe firms. These firms initially have no loyal base of customers, so there is no immediate sacrifice in revenues from loyal customers by aggressively pricing low to gain share. Later, once the fringe has a substantial market, there is an incentive to raise price to increase revenues from the loyal customers.

3. Dominant firm advertising is increasing. This is consistent with the fact that in this application advertising targets the fringe’s customers to try to get them to switch. Consequently, as the fringe grows, the dominant firm’s incentive to advertise increases.

4. Fringe advertising increases and then decreases. Initially, as the fringe price rises, the incentives to advertise also rise. On the other hand, since advertising targets the dominant firm’s customers, the fall in dominant firm share is a counterbalancing effect.

4.4. Comparison with the actual data

We compare the equilibrium paths from the previous subsection with the actual data of the firms. As a measure to summarize the fit between the actual and equilibrium levels for price and advertising for both players in the market, we use the root relative mean squared error (RRMSE) as in Chintagunta and Vilcassim (1994) and in Fruchter and Kalish (1997),

$$RRMSE = \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{Actual_t - Equilibrium_t}{Actual_t} \right)^2 \right)^{1/2},$$

where, in our case $T = 11$. The results are reported in Table 4.
From Table 4 we see that for AT&T the RRMSEs for both price and advertising spending are smaller than that for the fringe. These results indicate AT&T has adhered closer to the optimal solution in both price and advertising than the fringe. Examination of the actual data in Table 1, in comparison with the equilibrium solutions of Fig. 3, suggests two possible non-mutually-exclusive explanations for AT&T and fringe behavior in the 1980s:

1. AT&T and the fringe may be behaving in a risk averse fashion avoiding spending as much on advertising as the model suggests.
2. The fringe firms may be limited from increasing price (according to the model predictions) because of regulatory constraints and the desire to avoid consumer backlash from apparent opportunistic pricing policies.

5. Conclusion

In this paper we develop and analyze a dynamic game model in which an entrenched firm influences the entry of smaller competitors (termed as fringe entry) through its pricing and advertising policies. This issue would seem particularly pertinent in today’s context of the Internet and the World Wide Web.

Given a possible reaction of the fringe entry, either strategic or non-strategic, we present a new methodology to find time-invariant feedback Stackelberg equilibrium solutions in an implicit form by solving a set of two backward differential equations. The methodology relies on finding the relationship between the co-state variables and the state variable.

We apply the theoretical results to the case of AT&T facing the entry of MCI and Sprint during the period 1980–1990 and conclude with the following. At the beginning, AT&T finds it advantageous to take short-run revenues and, therefore, charges a high price. Later, once the fringe (composed from MCI and Sprint) has gained market, it is more advantageous for AT&T to lower the price. The advertising of AT&T working as an offensive strategy increases as the market of the fringe grows. The fringe’s advertising increases and then decreases. Initially, as the fringe price rises, the incentive to advertise rises. On the other hand, the fall of AT&T’s market share is a counterbalancing effect. The comparison with actual behavior indicates that AT&T has adhered closer to the optimal solution in both price and advertising than the fringe.

Acknowledgements

The authors thank Sean Acheson, Edward Park, and Alex Kolesch for research assistance. We also thank Gary Lilien and Suresh Sethi for helpful comments. We acknowledge the support of the Social Sciences and Humanities Research Council of Canada.
Appendix A. The derivations for the optimization problems

A.1. The fringe (follower) optimization problem

The current-value Hamiltonian of the competitive fringe is given by

\[ H^f(x, p^d(x), p^f(x), u^d(x), u^f(x), \lambda_f) = (p^f - c_f)x - (u^f)^2 + \lambda_f g(x, p^d(x), p^f(x), u^d(x), u^f(x)), \]  

(A.1)

where \( \lambda_f \) is the adjoint (or co-state) variable. The necessary optimality conditions for this problem are

\[ \frac{\partial H^f}{\partial p^f} = 0 \quad \text{and} \quad \frac{\partial H^f}{\partial u^f} = 0, \]

or

\[ x + \lambda_f \frac{\partial g(x, p^d, p^f, u^d, u^f)}{\partial p^f} = 0 \]  

(A.2)

and

\[ -2u^f + \lambda_f \frac{\partial g(x, p^d, p^f, u^d, u^f)}{\partial u^f} = 0. \]  

(A.3)

The adjoint equation, \( \dot{\lambda}_f = r\lambda_f - \frac{\partial H^f}{\partial x} - \frac{\partial H^f}{\partial p^d}(d_p^d/dx) - \frac{\partial H^f}{\partial u^d}(d_u^d/dx) \), is

\[ \dot{\lambda}_f = r\lambda_f - (p^f - c_f) - \lambda_f \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial p^d} \frac{d p^d}{d x} + \frac{\partial g}{\partial u^d} \frac{d u^d}{d x} \right), \]  

(A.4)

and the terminal condition is

\[ \lim_{t \to \infty} \lambda_f(t) e^{-rt} = 0. \]  

(A.5)

Denote the first- and second-order derivatives of \( g \) with respect to \( p^f \) and \( u^f \) by \( g_{p^f}, g_{u^f}, g_{p^f p^f}, g_{p^f u^f}, g_{u^f u^f} \). Also let \( p^{f^*} \) and \( u^{f^*} \) be the optimal price and advertising solution. A sufficient condition for a local optimum is that the Hessian matrix of \( H_f \), denoted by \( \tilde{H}_f \), is negative definite, i.e.,

\[ \tilde{H}_f = \begin{bmatrix} \lambda_f g_{p^f} & \lambda_f g_{p^f u^f} \\ \lambda_f g_{u^f} & -2 + \lambda_f g_{u^f u^f} & \end{bmatrix} < 0 \]  

(A.6)

for all \( (p^f, u^f) \) near \( (p^{f^*}, u^{f^*}) \).

Using condition (A.6), we may apply the implicit function theorem (see Apostol, 1979, p. 374) to the system Eq. (A.2) and (A.3) to arrive at the unique value of the control variables \( p^{f^*} \) and \( u^{f^*} \) in terms of the state and co-state variables, \( x \) and \( \lambda_f \), and the dominant firm’s control variables, \( p^d \) and \( u^d \),

\[ p^{f^*} = p^{f^*}(x, \lambda_f, p^d, u^d) \]  

(A.7)

---

6 This requires that the principal minors of matrix \( \tilde{H}_f \) alternate in sign, beginning with negative.
and

$$u^r = u^r(x, \lambda_f^r, p_d, u_d^r).$$  

(A.8)

In what follows we use (A.7) and (A.8) as the characterization of the competitive firm’s reaction. The equation

$$\dot{x} = g(x, p_d, p^r_d(x, \lambda_f^r, p_d, u_d^r), u_d^r, u^r (x, \lambda_f^r, p_d, u_d^r)) = \bar{g}(x, \lambda_f^r, p_d, u_d^r)$$

(A.9)

then becomes a constraint of the dominant firm’s optimization problem.

Next we analyze the optimal control problem of the dominant firm.

A.2. The dominant firm (leader) optimization problem

The current-value Hamiltonian of the dominant firm is,

$$H_d(x, p_d, u_d(x, \lambda_f^r, \lambda_d^r) = (p_d - c_d)(\bar{f}(x, \lambda_f^r, p_d, u_d^r) - x) - u_d^2$$

$$+ \lambda_d \bar{g}(x, \lambda_f^r, p_d, u_d^r),$$

(A.10)

where

$$\bar{f}(x, \lambda_f^r, p_d, u_d^r) = f(p_d, p^r_d(x, \lambda_f^r, p_d, u_d^r), u_d^r, u^r (x, \lambda_f^r, p_d, u_d^r))$$

(A.11)

and $\lambda_d$ is the adjoint (or co-state) variable. The necessary optimality conditions for this problem are $\partial H_d/\partial p_d = 0$ and $\partial H_d/\partial u_d = 0$, or, respectively,

$$\bar{f}(x, \lambda_f^r, p_d, u_d^r) - x + (p_d - c_d) \frac{\partial \bar{f}(x, \lambda_f^r, p_d, u_d^r)}{\partial p_d} + \lambda_d \frac{\partial \bar{g}(x, \lambda_f^r, p_d, u_d^r)}{\partial p_d} = 0$$

(A.12)

and

$$(p_d - c_d) \frac{\partial \bar{f}(x, \lambda_f^r, p_d, u_d^r)}{\partial u_d} - 2u_d + \lambda_d \frac{\partial \bar{g}(x, \lambda_f^r, p_d, u_d^r)}{\partial u_d} = 0.$$  

(A.13)

The adjoint equation, $\dot{\lambda}_d = r\lambda_d - \partial H_d/\partial x$, is

$$\dot{\lambda}_d = r\lambda_d + (p_d - c_d) \left(1 - \frac{\partial \bar{f}(x, \lambda_f^r, p_d, u_d^r)}{\partial x}\right) - \lambda_d \frac{\partial \bar{g}(x, \lambda_f^r, p_d, u_d^r)}{\partial x}$$

(A.14)

and the terminal condition is

$$\lim_{t \to \infty} \lambda_d(t)e^{-rt} = 0$$

(A.15)

Denote the first- and second-order derivatives of $\bar{f}$ with respect to $p_d$, $u_d$ by $\bar{f}_{p_d}, \bar{f}_{u_d}, \bar{f}_{p^2_d}, \bar{f}_{p_d u_d}, \bar{f}_{u^2_d}$ (denote the derivatives of $\bar{g}$ similarly). Also let $p_d^*$ and $u_d^*$ be the optimal price and advertising solution. A sufficient condition for a local optimum is that the Hessian matrix of $H_d$, denoted by $\bar{H}_d$, is negative
definite, i.e.,
\[
\tilde{H}_d = \begin{bmatrix}
2\tilde{f}_{p^d} + (p^d - c_d)\tilde{f}_{p^d p^d} + \lambda_d g_{p^d p^d} & \tilde{f}_{u^d} + (p^d - c_d)\tilde{f}_{p^d u^d} + \lambda_d g_{p^d u^d} \\
\tilde{f}_{u^d} + (p^d - c_d)\tilde{f}_{u^d p^d} + \lambda_d g_{u^d p^d} & (p^d - c_d)\tilde{f}_{u^d u^d} - 2 + \lambda_d g_{u^d u^d}
\end{bmatrix} < 0
\] (A.16)

for all \((p^d, u^d)\) near \((p^{d*}, u^{d*})\).

Eqs. (A.12) and (A.13) describe how the solution \((p^{d*}, u^{d*})\) depends on \((x; \lambda_d, \lambda_f)\). In particular, Eqs. (A.12) and (A.13) can be viewed as defining functions \(p^{d*} = p^{d*}(x; \lambda_d, \lambda_f)\) and \(u^{d*} = u^{d*}(x; \lambda_d, \lambda_f)\) (under conditions to be made more precise shortly).

A.3. The case: the fringe is price-taker and advertises

In this case, the necessary optimality conditions (A.2) and (A.3) are reduced to (A.3) and the sufficient condition (A.6) is reduced to the condition: 
\[-2 + \lambda_f g_{u^d u^d} < 0.\]

A.4. The case: the fringe is passive (price-taker and does not advertise)

The necessary optimality conditions for this problem become
\[
f(p^d, u^d) - x + (p^d - c_d)\frac{\partial f(p^d, u^d)}{\partial p^d} + \lambda_d \frac{\partial g(x, p^d, u^d)}{\partial p^d} = 0
\] (A.17)

and
\[
(p^d - c_d)\frac{\partial f(p^d, u^d)}{\partial u^d} - 2u^d + \lambda_d \frac{\partial g(x, p^d, u^d)}{\partial u^d} = 0.
\] (A.18)

The adjoint equation becomes
\[
\dot{\lambda}_d = r\lambda_d + (p^d - c_d) - \lambda_d \frac{\partial g(x, p^d, u^d)}{\partial x}
\] (A.19)

and the terminal condition is
\[
\lim_{t \to \infty} \lambda_d(t)e^{-rt} = 0.
\] (A.20)

The sufficient condition for a local optimum is that the Hessian matrix of the corresponding Hamiltonian, \(\tilde{H}_d\), is negative definite, i.e.,
\[
\tilde{H}_d = \begin{bmatrix}
2f_{p^d} + (p^d - c_d)f_{p^d p^d} + \lambda_d g_{p^d p^d} & f_{u^d} + (p^d - c_d)f_{p^d u^d} + \lambda_d g_{p^d u^d} \\
f_{u^d} + (p^d - c_d)f_{u^d p^d} + \lambda_d g_{u^d p^d} & (p^d - c_d)f_{u^d u^d} - 2 + \lambda_d g_{u^d u^d}
\end{bmatrix} < 0
\] (A.21)

for all \((p^d, u^d)\) near \((p^{d*}, u^{d*})\).
Appendix B. Proof of the Main Theorem

Consider the necessary, (A.12) and (A.13), and sufficient (A.16) conditions for local optimality for the optimal control problem of the leader (dominant firm). In particular, if condition (A.16) holds in a neighborhood of \((p^d^*, u^d^*)\), we may apply the implicit function theorem (see Apostol, 1979, p. 374) to Eqs. (A.12) and (A.13) to arrive at the unique values of the control variables \(p^d^*\) and \(u^d^*\) in terms of the state and co-states, \(x, \lambda_d, \text{ and } \lambda_f\),

\[
\begin{align*}
p^{d^*} &= p^{d^*}(x, \lambda_d, \lambda_f) \quad \text{and} \quad u^{d^*} = u^{d^*}(x, \lambda_d, \lambda_f). \quad (B.1)
\end{align*}
\]

Considering (10), we obtain (11) and in particular we have,

\[
\begin{align*}
p'(x)\dot{x} &= \dot{\lambda}_d \quad \text{and} \quad \Phi'(x)\dot{x} = \dot{\lambda}_f. \quad (B.2)
\end{align*}
\]

Consider now the necessary (A.2) and (A.3) and sufficient (A.6) conditions for local optimality for the optimal control problem of the follower (competitive fringe). In particular, if condition (A.6) holds in a neighborhood of \((p^f^*, u^f^*)\), we obtain the unique values of the control variables \(p^f^*\) and \(u^f^*\) as defined in (6) and (7), (again by applying the implicit function theorem). Substituting (10) in (6) and (7), we obtain the unique values in (12). Now the theorem follows immediately from (A.4) and (A.14), after substituting in Eqs. (B.1), (10), (B.2), (12) and (8), recognizing the corresponding terminal conditions, (A.5) and (A.15). \(\square\)

References


Whiteside, T., 1951. Where are they now?, New Yorker (February 17), 39–58.