The Many-Player Advertising Game

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This study extends the time-variant closed-loop strategy of Fruchter and Kalish (1997) to the \( n \)-player advertising game. We demonstrate that solving an \( n \)-player game using 2 players results in overadvertising.

\( \text{Differential Games; Noncooperative; Nash Equilibrium; Marketing-Advertising} \)

1. Introduction
This study extends the time-variant closed-loop strategy of Fruchter and Kalish (1997) to the \( n \)-player advertising game. We show using the 2-player results for an \( n \)-player situation results in overadvertising.

2. The Many-Player Advertising (AD) Game
Consider the dynamic system given by the “combat equations,” known as the Lanchester model, for the \( n \)-player case,

\[
\dot{x}_k(t) = \rho_k u_k(t) - x_k(t) \sum_{j=1}^{n} \rho_j u_j(t), \quad x_k(0) = x_0^k, \quad k = 1, \ldots, n, \tag{1a}
\]

where \( x_k(t), u_k(t) \geq 0, \forall k, \) and

\[
\sum_{j=1}^{n} x_j(t) = 1. \tag{1b}
\]

The state variable \( x_k(t), k = 1, \ldots, n, \) represents the market share of firm \( k \) at time \( t \), and the control variable \( u_k(t), k = 1, \ldots, n, \) is the square root of firm \( k \)'s marketing expenditures at time \( t \). The constant \( \rho_k, k = 1, \ldots, n, \) measures the effectiveness of advertising efforts of firm \( k \) to attract competitors' sales.

Consider now the following standard discounted profit \( n \)-objective functions for the competitors

\[
\prod_k (u_1, \ldots, u_n) = \int_0^\infty [q_k x_k(t) - r_k u_k^2(t)] e^{-\rho t} dt, \quad k = 1, \ldots, n. \tag{1c}
\]

The constant \( q_k, k = 1, \ldots, n, \) represents the gross profit rate; the term \( \rho \) is the constant discount rate; and \( r_k, k = 1, \ldots, n, \) represents the effectiveness of advertising buying power (perhaps, each firm can arrange a different discount on advertising). Usually, \( r_k = 1, k = 1, \ldots, n. \)

The objective is to find closed-loop strategies \( u_k^*, k = 1, \ldots, n, \) satisfying the following inequality conditions:

\[
\prod_k (u_1^*, \ldots, u_n^*) \geq \prod_k (u_1^*, \ldots, u_{k-1}^*, u_k^*, u_{k+1}^*, \ldots, u_n^*), \quad k = 1, \ldots, n, \forall u_k. \tag{2}
\]

In other words, we want to find a strategy \( (u_1^*, \ldots, u_n^*) \) that is a closed-loop Nash equilibrium of the differential game (1a–c).

The class of closed-loop strategies is characterized by the requirement that the control at each point in time be a function of both time, state, and initial state.
3. Nash Equilibrium Closed-Loop Strategies

The Main Theorem. Consider the n-player advertising game (1a–c). Let

\[ u^*_k = \frac{1}{2} r_k^{-1} \rho_k q_k \varphi^*(1 - x_k), \quad k = 1, \ldots, n \]  

(3)

where \( x_k = x_k(t) \) is as in (1a–b) and \( \varphi = \varphi(t) \) satisfies the following two-point boundary value problem,

\[
x^*_k = \frac{1}{2} \left[ \rho^*_k r_k^{-1} q_k (1 - x^*_k) - x_k^* \sum_{j=1}^{n} \rho^*_j r_j^{-1} q_j (1 - x^*_j) \right] e^{\rho^* \phi}, \\
\phi = \frac{1}{2} \left[ \sum_{j=1}^{n} \rho^*_j r_j^{-1} q_j (1 - x^*_j) \right] e^{\rho^* \phi} - e^{-\rho^* \phi}, \\
\phi(0) = 0.
\]

(4)

Then \((u^*_1, \ldots, u^*_n)\) forms a Nash equilibrium closed-loop strategy of the above differential game, i.e., it satisfies Condition (2).\(^1\)

Proof. Available from the author on request.

Remark 1. Consider the Main Theorem where

\[ \rho_k = \rho_2, \quad q_k = q_2, \quad r_k = r_2 \quad \text{for } k \geq 2. \]

(5)

Let

\[ \phi(t) e^{\rho^* t} = \varphi(x^*_k), \]

(6)

where \( x^*_k = 1 - \sum_{i=2}^{n} x^*_i \) and \( x^*_k \) is as in (4). Then \( \varphi \) satisfies

\[
\varphi(x^*_k) \varphi'(x^*_k) \left[ \rho^*_k r_k^{-1} q_k (1 - x^*_k)^2 - \rho^*_k r_k^{-1} q_k (n - 2 + x^*_k) x^*_k \right] = \varphi(x^*_k) \left[ \rho^*_k r_k^{-1} q_k (1 - x^*_k) + \rho^*_k r_k^{-1} q_k (n - 2 + x^*_k) \right] + 2 \rho^*_k \varphi(x^*_k) - 2,
\]

\[ \lim_{t \to \infty} \varphi(x^*_k(t)) e^{-\rho^* t} = 0. \]

(7)

Equation (7) follows from (4), (5), (6), (1b), and the fact that taking the time derivative from both sides of (6) leads to \( \dot{\varphi}(t) e^{\rho^* t} + \rho^* \varphi(x^*_k) = \varphi'(x^*_k) x^*_k. \)

Remark 2. Considering Remark 1, the Nash equilibrium closed-loop strategy in (3) has the form

\[ u^*_k = \frac{1}{2} r_k^{-1} \rho_k q_k \varphi^*(1 - x_k), \quad k = 1, \ldots, n \]

(8)

where \( \varphi \) satisfies (7).

Remark 3. Consider the symmetric case

\[ \rho_k = \rho_1, \quad q_k = q_1 \quad \text{and } r_k = r_1 \quad \text{for } k \geq 1, \]

and zero discount. Then (7) can be solved analytically and (8) becomes, for any \( n \),

\[ u^*_k = \left( \frac{1}{2} r_1^{-1} \rho_1 q_1 \right)^2 (1 - nx^*_k)^{-2+2/n} \]

\[ \times \left( \frac{2(1 - nx^*_k)^{2-2/n}}{r_1^{-1} \rho_1 q_1 (n - 1) + m} (1 - x_k)^2, \right) \]

(10)

where \( m \) is an arbitrary constant.

Remark 4. One can check that the strategy in (10) is monotonically decreasing with \( n \), for \( n \geq 2 \).

4. Marketing Implication

In the following we want to assess the advertising expenditures of an oligopolist in using an \( n \)-player game, over the 2-player game. To simplify the comparison we consider a situation when all firms satisfy the “symmetric case,” as in Remark 3.

If player \( k \) is using the 2-player game then all the other players should be aggregated into one rival and the advertising expenditure of this player is given by Formula (10) for \( n = 2 \). In contrast, if player \( k \) is using the \( n \)-player game, \( n > 2 \), then each player is modeled by choosing an advertising strategy as in Formula (10), but in this case \( n > 2 \). According to Remark 4, we conclude with the following.

Corollary. In an oligopoly competition, using a 2-player advertising game instead of an \( n \)-player, by aggregating all other players into one rival, leads to overadvertising.

5. Conclusion

This study finds a time-variant closed-loop solution to the \( n \)-player advertising game. The following contributions are obtained:

\(^1\) The algorithm of constructing the closed-loop strategy defined in (3) and (4) is similar to the case of 2-player Ad game of Fruchter and Kalish (1997).
It extends the results of Fruchter and Kalish (1997) to the $n$-player advertising game.

It demonstrates that solving an $n$-player game using 2 players results in overadvertising. The value of a closed-loop strategy is that such a strategy allows a manager to adjust to changing market conditions in an optimal way. This enhances the ability to deal effectively with dynamic competition in the development of advertising strategies.

Given the attractiveness of closed-loop strategies and the importance of modeling an oligopolistic competition, the present study fills a gap in the existing literature. We hope that the new approach and analysis of this paper will stimulate further research in the competitive marketing strategy literature. An obvious extension would be to consider a growing market, that is, a market that is not saturated.²

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Reference