Theory and Methodology

A dual control problem and application to marketing

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Abstract

The general approach to adaptive and dual control is to formulate an optimal stochastic control problem. However, for such an approach only mathematical representations of the solution are available which allow little insight into the structure of the optimal controller. Here, an alternative deterministic approach is presented based upon determining a control in which a disturbance attenuation function remains bounded for all allowable ($L_2$ functions) disturbances. The disturbance attenuation function is composed of the ratio of an $L_2$ function of the desired outputs over an $L_2$ function of the disturbance inputs. This disturbance attenuation problem is converted to a differential game. For this game, the optimal control law, in a closed-form, is obtained by performing a minmax operation with respect to a quadratic cost function subjected to a bilinear system. The resulting controller is time-varying and depends nonlinearly on the state and the parameter estimates vector, and on an associated Riccati-type matrix. We provide insights into the structure of the resulting dual controller and illustrate the method by two examples. One of the examples is an application to marketing, to set promotional spending of a company, considering that the effect of promotional effort on sales is unknown. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Considering a linear system in the presence of disturbance and parameter uncertainties, we present a game-theoretic approach to a worst case design problem. Formulating the problem as a deterministic game, the best control for the worst case initial states and measurement disturbances is found. The objective is to use a game-theoretic approach to design an optimal adaptive controller. Such a controller can be regarded as composed of two parts: a nonlinear estimator and a feedback regulator.

Presently, as summarized in Aström and Wittenmark (1989), Whittle (1991) and Basar and Bernhard (1991), the method of finding adaptive controllers is based upon the certainty equivalence principle. In contrast to this kind of adaptive controller, we want to add an active learning feature to the controller. In this case, the optimal
controller will be a dual controller. It is a compromise between maintaining optimal control and small estimation errors. Note that such a controller, with the property of learning and control, was first proposed by Feldbaum (1965). A general approach for this problem is presented, see Aström and Wittenmark (1989), by formulating it as a stochastic control problem. In stochastic optimal control problems, in general, see for example the review paper of Neck (1984), it will not be possible to handle the dual nature of the problem, i.e., to obtain jointly optimal statistical identification and control of the system. Bryson and Ho (1975) made an attempt to apply a deterministic game-theoretic formulation to a type of dual control problem. However, it was shown that even a very simple problem exhibits very complex features.

While it is natural to model the unknowns as random processes and find the control to minimize the expected value of the performance index, so little is known about the statistics that an adequate stochastic formulation is not available. In this paper, an alternative approach is taken by formulating a deterministic disturbance attenuation problem. In the disturbance attenuation problem, a controller is sought such that disturbance attenuation is bounded for all $L_2$ disturbance inputs. The disturbance attenuation function is an input–output representation constructed as the ratio of an $L_2$ function of the performance outputs over an $L_2$ function of the input disturbances, cf. Basar and Bernhard (1991). This disturbance attenuation problem is converted into a differential game where the quadratic cost criterion is constructed to be minimized by the control and maximized by the input disturbances. A game-theoretic approach to the linear quadratic regulator problem has been the subject of intense research efforts, see Whittle (1981) and Basar and Bernhard (1991).

The main purpose of this paper is to formulate a game for a special type of dual control problem and gain insights into some features that do not arise for linear systems. Adopting the game-theoretic formulation, we find a method to solve the problem analytically, in a closed form.

In Section 2, we consider a linear system with unknown initial conditions, measurement disturbances and without process disturbances, but with uncertainty, in the coefficient of the control. The dynamics of the uncertainty are described by a linear differential equation. To solve for a dual controller, i.e., a controller that estimates the uncertainty and regulates the system, the linear system is transformed into a bilinear system, where the new state is composed of the original state and the uncertain parameter. Formulating the game problem, where the disturbances and initial states are the adversaries, we are maximizing the cost function with respect to initial values and the measurement disturbances and minimizing with respect to the control. The order of the extremization with respect to the adversaries is important to determine controller strategies that are explicit functions of the measurement sequence. This is done by first maximizing the game cost criterion with respect to the process disturbance and, in our case, the unknown condition $\xi(0)$. The resulting controller is time-varying and depends non-linearly on the state and parameter estimate vectors and their associated Riccati matrix. An insight into the structure of the dual controller is given in Section 5.

The dual controller, developed in this paper, can be applied to find marketing strategies. In Section 6, we illustrate such an application to set promotional spending of a company, considering that the effect of promotional effort on sales is unknown. Several marketing researchers dealt with measuring the effect of promotion (as advertising) on sales, among them, Clarke (1976), Banks (1965), Rao (1970), Pekelman and Tse (1980), Little (1979), Eastlack and Rao (1986), Mahajan and Muller (1986), Winer and Moore (1989). However, the method proposed in this paper offers an adaptive control scheme that has the ability to simultaneously estimate the effects on line when new data appear and to regulate the promotional setting by diminishing the market noise and unknowns effects. This is especially useful in new products, when data on sales rates are not available on how to use econometric methods to measure the promotional spending effect on the sales rate. The idea of applying adaptive control theory...
to managerial problems has been used by many other researchers, among them, Deissenberg and Stoppler (1983).

2. Problem formulation

Consider the continuous dynamical system with parameter uncertainty in the coefficient of the control given by

$$\dot{x}(t) = A_1(t)x(t) + k(t)u(t),$$

$$\dot{k}(t) = A_2(t)k(t).$$

(1a)

(1b)

Here, $x, k \in \mathbb{R}^n$, where $x$ is the state, and $k$ is a vector of unknown parameters in the control coefficient, modeled by Eq. (1b), $A_1, A_2 \in \mathbb{R}^{n \times n}$ are given time-varying matrices, and $u \in \mathcal{R}$ is the input control, respectively. A linear measurement of the state, $y \in \mathbb{R}^m$, containing an additive disturbance, $v \in \mathbb{R}^m$, is also assumed:

$$y(t) = H(t)x(t) + v(t),$$

(1c)

where $H \in \mathbb{R}^{m \times n}$ is a time-varying matrix.

Define the measurement history up to $t$ as

$$Y_t = \{y(s) : 0 \leq s \leq t\}.$$

The admissible control is restricted to be a function of only the measurement history $Y_t$. Let

$$\xi^T = [x^T \ k^T].$$

Then (1a)–(1c) can be described by the following continuous bilinear system:

$$\dot{\xi} = A\xi + Bu\xi,$$

$$y = E\xi + v,$$

(2a)

(2b)

where

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

$I$ is the identity matrix, and $E = [H \ 0]$.

Now, consider the following performance index, motivated by the disturbance attenuation problem, see Basar and Bernhard (1991) and the 'stress' performance index of Whittle (1990, 1991):

$$J = J(u, v, \xi(0))$$

$$= \frac{1}{2} \left( ||\xi(T)||_{Q_2}^2 + \frac{1}{\theta} ||\xi(0) - \hat{\xi}_0||_{P_0}^2 \right)$$

$$+ \int_0^T \left( u^2 + \frac{1}{\theta} ||v||_{F-1}^2 \right) dt. \quad (3)$$

Here, $\xi_0^T = [x_0^T \ k_0^T]$, $\theta$ is a negative constant, $P_0$ and $V = V(t)$ are positive definite symmetric matrices and $Q_2$ is a constant nonnegative definite symmetric matrix, of compatible dimensions, respectively.

Our objective is to find a control which is a function of the measurement history up to $t$, $u(Y_t) \in L_2[0, T]$, which will be the best adaptive control for the worst case disturbance $v \in L_2[0, T]$ and worst case initial condition $\xi(0)$. Such a control will minimize the payoff functional for worst case disturbance and initial state by estimating the uncertainty. As will be shown and as expected, since the system is bilinear, control and estimation cannot be designed separately. The estimation is affected by the control, which can perform the dual roles of regulation and learning (the uncertainty $k$). Therefore, the adaptive controller will be a dual controller, it affects the acquisition of information as well as the regulation of the process.

Considering this problem as a deterministic game, the optimal dual controller is obtained by performing the following minimax operations:

$$\min_{u} \max_{\xi(0)} \max_{v} J.$$

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$^1$The disturbance attenuation problem is to find the admissible control $u(Y_t)$ such that the attenuation function, defined as the ratio of quadratic functions of the output performance measures (represented here by the states and control) to the quadratic functions of the input disturbances including the initial state, is bounded as

$$\frac{||\xi(T)||_{Q_2}^2 + \int_0^T u^2 dt}{||\xi(0) - \hat{\xi}_0||_{P_0}^2 + \int_0^T ||v||_{F-1}^2 dt} < \frac{1}{\theta} \quad \forall v \in L_2[0, T], \ \xi(0) \in \mathbb{R}^m, \ \theta < 0.$$
subjected to Eqs. (2a) and (2b). In this game, the control \( u \) wants to minimize exactly what the adversaries \( \xi(0) \) and \( v \) want to maximize. The maximization operation belongs to the estimation process and the minimization operation gives us the optimal desired dual control law. The desired solution is obtained in two steps:

(i) We find the optimal open-loop strategy. For this step we do not need to separate the maximization operations.

(ii) We use the open-loop strategy for finding the desired closed-loop strategy. In this step we do not need to separate the maximization operations.

Remark 1. The separation and the order of the maximization problem are important for our final goal to find a closed-loop strategy that is an explicit function of the measurement. If the cost criterion (3) is maximized with respect to \( \xi(0) \), assuming \( u \) and \( y \) are given, then a state reconstruction scheme is obtained. Given this state estimate along with \( u \) and \( y \), the measurement disturbance can be computed. This state reconstruction scheme is found in Section 3.1. In Section 3.4, the maximization with respect to the residual of the state reconstruction is performed resulting in a nonlinear optimal control in \( u \).

Note that the cost criterion for the continuous deterministic game reminds us of the criterion of the Linear-Exponential-Gaussian stochastic control problem cf. Whittle (1981) and Besoussan and Schuppen (1985).

3. Estimation of the uncertainty

3.1. Maximization with respect to the initial condition \( \xi(0) \)

Considering the standard variational problem for the cost \( J \) defined in (3) subjected to the dynamical system (2a) and (2b), with fixed arbitrary strategies \( v \) and \( u \), the following two-point boundary value problem can be obtained:

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\lambda}
\end{bmatrix} = \begin{bmatrix}
A + Bu & 0 \\
-E^T V^{-1}Ey & -(A + Bu)^T
\end{bmatrix} 
\begin{bmatrix}
\xi \\
\lambda
\end{bmatrix} + \begin{bmatrix}
0 \\
E^T V^{-1}y
\end{bmatrix},
\]

\[\xi(0) = \tilde{\xi}_0 - P_0 \tilde{\lambda}(0), \quad \tilde{\lambda}(T) = 0 Q_T \tilde{\xi}(T). \tag{4}\]

where \( \tilde{\lambda} \) represents the Lagrange multiplier (adjoint variable) and \( y \) is defined in (2b). Since \( P_0 \) is positive definite and \( \theta < 0 \), the necessary condition (4), for the initial condition, is also sufficient and it is a local maximum for the cost function \( J \).

For solving problem (4), we use the sweep method as in Bryson and Ho (1975).

3.2. Derivation of estimator structure

Let

\[\xi(t) = \tilde{\xi}(t) - P(t) \tilde{\lambda}(t), \quad t \in [0, T], \tag{5a}\]

where \( \tilde{\xi} \) and \( P \) are to be determined. Differentiating (5a) and considering (4) and again (5a), we obtain

\[
\dot{\xi} = (A + Bu) \tilde{\xi} - (A + Bu) P \tilde{\lambda}
\]
\[= \dot{\tilde{\xi}} - \dot{P} \tilde{\lambda} - P \dot{\lambda}
\]
\[= \dot{\tilde{\xi}} - \dot{P} \tilde{\lambda} - [E^T V^{-1} E (\dot{\tilde{\xi}} - P \tilde{\lambda})
\]
\[- (A + Bu)^T \lambda + E^T V^{-1} y].\]

Rearranging this equation, we obtain

\[
\dot{\xi} - (A + Bu) \tilde{\xi} + PE^T V^{-1} (E \tilde{\xi} - y)
\]
\[= \dot{P} + PE^T V^{-1} EP - P (A + Bu)^T
\]
\[- (A + Bu) \tilde{\lambda}. \tag{5b}\]

In conclusion, for any \( t \in [0, T] \), an estimator can be chosen so that

\[
\dot{\tilde{\xi}} = (A + Bu) \tilde{\xi} + PE^T V^{-1} \tilde{v}, \quad \tilde{\xi}(0) = \tilde{\xi}_0, \tag{6}\]

where \( \tilde{v} = y - E \tilde{\xi} \) and \( P \) satisfies

\[
\dot{P} = (A + Bu) P + P (A + Bu)^T - PE^T V^{-1} EP, \quad P(0) = P_0. \tag{7}\]
The estimator, defined by (6) and (7), has a role similar to the Kalman filter.

3.3. Cost criterion as a function of filter variables

In the following, we want to rewrite $J$ in terms of the filter variables, $\bar{\xi}$ and $\bar{v}$:

Considering the first-order necessary condition (4), we obtain

$$J = \frac{1}{2} \left[ \|\bar{x}(T)\|_{Q_x}^2 + \frac{1}{\theta} \|\bar{\lambda}(T)\|_{P(T)}^2 \right] + \int_0^T \left( u^2 + \frac{1}{\theta} \|\bar{v}\|^2_{\bar{V}^{-1}} \right) dt.$$ 

Now consider the zero sum

$$0 = \frac{1}{2\theta} \left[ \bar{\lambda}^T P \bar{\lambda} \right]_{T_0}^{T} - \int_0^T \frac{d}{dt} \left( \bar{\lambda}^T P \bar{\lambda} \right) dt.$$ 

Adding this zero sum to the above $J$, and considering Eqs. (2b), (4)–(5b) and (7), we obtain, after some manipulations,

$$J = \frac{1}{2} \left[ \|\bar{x}(T)\|_{Q_x}^2 + \frac{1}{\theta} \|\bar{\lambda}(T)\|_{P(T)}^2 \right] + \int_0^T \left( u^2 + \frac{1}{\theta} \|\bar{v}\|^2_{\bar{V}^{-1}} \right) dt,$$

$$= \frac{1}{2} \left[ \|\bar{x}(T)\|_{Q_x(I + \theta P(T)Q_x)}^2 + \int_0^T \left( u^2 + \frac{1}{\theta} \|\bar{v}\|^2_{\bar{V}^{-1}} \right) dt \right],$$

where

$$S_T = Q_T[I + \theta P(T)Q_T]^{-1}.$$ 

Note that we assume that $I + \theta P(T)Q_T$ is a non-singular matrix.

3.4. Maximization with respect to $\bar{v}$

Repeating now the variational method to the cost function (8a) and (8b) subjected to Eq. (6), for fixed arbitrary $u$, the following two-point boundary value problem is obtained:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A + Bu & -\theta PE^T V^{-1} EP \\ 0 & -(A + Bu)^T \end{bmatrix} \begin{bmatrix} \xi \\ \lambda \end{bmatrix},$$

$$\dot{\xi}(0) = \xi_0, \quad \dot{\lambda}(T) = S_T \bar{\xi}(T),$$

where $\dot{\lambda}$ is the Lagrange multiplier. Note that here the Hamiltonian is

$$H = \frac{1}{2} \left( u^2 + \frac{1}{\theta} \|\bar{v}\|^2_{\bar{V}^{-1}} \right) + \dot{\lambda}^T \left[ (A + Bu)\bar{\xi} + PE^T V^{-1} \bar{v} \right].$$

Since $V$ is positive definite and $\theta < 0$, $\partial^2 H/\partial \bar{v}^2 = (\partial V)^{-1}$ is negative definite and therefore the necessary condition $\dot{v} = -\theta EP\dot{\lambda}$, is also sufficient, it is a local maximum for $J$.

For solving (9), we use again the sweep method and define a new variable

$$\bar{\xi} = \bar{\xi} + \theta P\dot{\lambda}, \quad t \in [0, T].$$

For finding $\bar{\xi}$, we differentiate (10) and consider (9) and (10), and obtain

$$\dot{\xi} = (A + Bu)\bar{\xi} - \theta PE^T V^{-1} EP \dot{\lambda},$$

$$= (A + Bu)\bar{\xi} + \theta [A + Bu] P - P [E^T V^{-1} EP] \dot{\lambda},$$

$$= \dot{\xi} + \theta [P\dot{\lambda} + P\dot{\lambda}],$$

Considering this equation, we have

$$\dot{\xi} - (A + Bu)\bar{\xi} = \theta \left[ -P + P(A + Bu)^T + (A + Bu)P - PE^T V^{-1} EP \right] \dot{\lambda}.$$ 

Considering (7) and (11) we obtain that the new variable has to satisfy

$$\dot{\bar{\xi}} - (A + Bu)\bar{\xi}, \quad \bar{\xi}(0) = \xi_0 - \theta P_0 \dot{\lambda}(0).$$

The cost function in (8a) and (8b) in terms of the new variable will become
\[ J = \frac{1}{2} \left( \| \hat{\xi}(T) \|_{Q_T}^2 + \frac{1}{\theta} \| \hat{\xi}(0) - \hat{\xi}_0 \|_{P_0^{-1}}^2 + \int_0^T u^2 \, dt \right). \]  
(13)

4. Determination of the optimal dual control law

4.1. Minimization with respect to \( u \)

For performing the last step of the variational problem, we have to consider the cost function in (13) subjected to Eq. (12). In this case, the following necessary conditions are obtained:

\[ \dot{\eta} = -(A + Bu)^T \eta, \quad \eta(T) = Q_T \hat{\xi}(T), \]  
(14)

\[ u = -\eta^T B \hat{\xi}, \]  
(15)

where \( \eta \) represents the Lagrange multiplier (adjoint variable).

The Hamiltonian in this case is

\[ H = \frac{1}{2} u^2 + \eta^T (A + Bu) \hat{\xi}, \]  

and \( \partial^2 H/\partial u^2 = 1 > 0 \), therefore the necessary condition (15) is also sufficient, and it is a local minimum for the cost function \( J \).

Another way to write the identity (8b) can be

\[ S_T = Q_T [I - \theta P(T) S_T]. \]

Considering this identity and the boundary condition of the second equation in (9), and (10) at \( t = T \), we obtain

\[ \hat{\lambda}(T) = S_T \hat{\xi}(T) = Q_T \{ I - \theta P(T) S_T \} \hat{\xi}(T) = Q_T \{ \hat{\xi}(T) - \theta P(T) \hat{\lambda}(T) \} = Q_T \hat{\xi}(T). \]  
(16)

Comparing now the second equation in (9), where the boundary condition is in the form (16), with the condition (14), we conclude with

\[ \eta = \hat{\lambda}. \]  
(17)

Considering (17) and (12) will become

\[ \hat{\xi} = (A + Bu) \hat{\xi}, \quad \hat{\xi}(0) = \hat{\xi}_0 - \theta P_0 \eta(0). \]  
(18)

Remark 2. Note that the special definition in (10), of choosing \( \theta P \) for the coefficient of \( \hat{\lambda} \), is crucial and essential, and is the key in solving the problem in this paper. Due to this definition, first, we reduce the complexity of the computation in the minimization process (since we do not have new constraint equations, i.e., an additional Riccati equation, as usually by the standard method, see condition (11)). Secondly, the identity (17) is a result of this definition.

4.2. The open-loop optimal strategy

The open-loop optimal control law is obtained by Eq. (15), where \( (\eta, \hat{\xi}) \) is a solution of the two-point boundary problem obtained from (14) and (18) by substituting (15) for \( u \), i.e.,

\[ \dot{\eta} = -(A - B \eta^T B \hat{\xi})^T \eta, \quad \eta(T) = Q_T, \]  
\[ \hat{\xi} = (A - B \eta^T B \hat{\xi}) \hat{\xi}, \quad \hat{\xi}(0) = \hat{\xi}_0 - \theta P_0 \eta(0). \]  
(19)

An analytical solution to this problem is illustrated by an example in Section 5. For more complicated cases, one can use a numerical method, like the shooting method, cf. Ascher et al. (1988) and Lambert (1973). Denote the solution in the following form:

\[ u_{ol} = \varphi(\hat{\xi}_0, P_0, T, t), \quad t \in [0, T], \]  
(20)

where the subscript denotes open loop. As mentioned in the next section, we illustrate how to find \( \varphi \) explicitly. Therefore, in the sequel we can regard \( \varphi \) as a known function.

Without any further information (at this stage we do not need Eqs. (6) and (7)), Eq. (20) is the best possible control. In the following, we will show how this open-loop strategy, together with the estimator Eqs. (6) and (7), can be used for finding the feedback solution which will depend on the actual measurements.

4.3. The closed-loop dual controller

Consider Eq. (20), where \( \varphi \) is known and calculated ‘off line’ by (15) and (19), as presented
above, where the initial conditions are $\hat{\xi}(t)$ and $P(t)$ rather than $\hat{\xi}_0$ and $P_0$. Also, the final time is the time to go,

$$T_{go} = T - t,$$

rather than $T$. Then the optimal feedback controller will be

$$u^*(t) = \begin{cases} u_{ol}, & t = 0, \\ \phi(\hat{\xi}(t), P(t), T_{go}, t), & t \neq 0. \end{cases}$$

The real time initial conditions $\hat{\xi}(t)$ and $P(t)$ are estimated from Eqs. (6) and (7). Note that the initial measurement $y(0)$ is found by using the input $u^*(0) = u_{ol}$, known in advance.

Next we illustrate by an example how one can find the control law (22).

5. Interpretation of the results

In Fig. 1, we find a schematic interpretation of the results presented here. The control $u^*$ is used both for estimating $\hat{\xi} = [\hat{x}^T \hat{k}^T]$ and regulating the system Eqs. (1a) and (1b). For this reason, $u^*$ is called the dual controller. Starting at $t = 0$, $u^*$ is applied to the system Eqs. (1a) and (1b) for finding the current measurement $y(0^+)$, which together with the input $u^* = u_{ol}(0^+)$ (calculated in advance on ‘off line’) is applied to the system (6) and (7) for estimating $\hat{\xi}(t)$, $P(t)$, in real time, by the ‘Estimation Box’. The desired time-varying feedback control law $u^*(t)$ follows by using the new data of ‘initial values at instant $t^*$’, $\hat{\xi}(t)$, $P(t)$, instead of the initial values $\hat{\xi}_0$, $P_0$, and the time to go $T_{go}$, instead of the final time $T$, in the ‘Regulation Box’.

6. A marketing application

The particular case, where (1a)–(1c) is a scalar system, and furthermore $A_1 = A_2 = 0$, $x(t)$ is scalar and $k$ is constant but unknown, can be useful to conduct a company’s marketing operations such as promotional spending. Company sales are functions of promotional spending, but the relation changes with time. The state $x(t)$ denotes the sales rate of a company at time $t$, and $u(t)$ the promotional effort at time $t$. It is assumed that promotional effort $u(t)$ represents the square root of the company’s promotional spending; this is a common assumption in the literature of advertising models, cf., Erickson (1992) and Fruchter and Kalish (1997). In most of the cases, and especially for new products, the effect of promotional effort on sales is unknown. We represent it by $k$. Eq. (1a) describes the dynamic relationship between the change in sales rate and the promotional effort and its effect on sales. Eq. (1c) takes into account the market noises, $v$, in measuring the sales rate at time $t$, as $y(t) = hx(t) + v$, where $h$ is constant. In other words, $v$ represents sales rate which is independent of the control $u$.

A graphical representation on how the dual controller can be applied to set promotional spending is described in Fig. 2. We start with deriving the optimal promotional spending using the ‘Regulation Box’ for $t = 0$. This amount is applied to the system of equation (that is, to the dynamic relationship between the rate of change in sales and the promotional effort and its effect on sales) to find the current sales measurement. The resulting measurements of sales rate together with the promotional rate just calculated are applied to the ‘Estimation Box’ to re-estimate the state of the system (sales rates and the effect of the promotional rate) in real time. Then set the promotional...
rate in real time by using the ‘Regulation Box’ and the cycle is repeated.

Next we illustrate the method developed in the previous sections to the above marketing problem.

Consider system (2a) and (2b) where
\[ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]
and
\[ E = [h \ 0]. \tag{23} \]

Assume also that \( QT \) in the cost function is given by
\[ QT = \begin{pmatrix} q_T & 0 \\ 0 & 0 \end{pmatrix}. \tag{24} \]

In (23) and (24) \( h \) and \( q_T \) are constants that can be determined from the history of a similar product situation.

For finding the function \( \varphi \) in (20), we use Eqs. (14), (15) and (18) (or (15) and (19)), with the initial values \( \tilde{z}_0, P_0 \).

Let \( \tilde{z} = [\tilde{x} \ \tilde{k}] \). In terms of \( \tilde{x} \) and \( \tilde{k} \), Eqs. (14), (15) and (18) will become
\[ \dot{\eta}_1 = 0, \quad \eta_1(T) = f, \tag{25} \]
\[ \dot{\eta}_2 = -u \eta_1, \quad \eta_2(T) = 0, \tag{26} \]
where
\[ f = q_T \tilde{x}(T), \tag{27} \]
\[ u = -\tilde{k} \eta_1 \tag{28} \]
and
\[ \dot{\tilde{x}} = u \tilde{k}, \]
\[ \tilde{x}(0) = \hat{x}_0 - \theta[p_{11}(0)\eta_1(0) + p_{12}(0)\eta_2(0)], \tag{29} \]
\[ \dot{\tilde{k}} = 0, \]
\[ \tilde{k}(0) = \hat{k}_0 - \theta[p_{21}(0)\eta_1(0) + p_{22}(0)\eta_2(0)]. \tag{30} \]

Integrating (25) we obtain
\[ \eta_1(t) = f \tag{31} \]
and
\[ \ddot{k}(t) = \ddot{k}(0). \]  \hfill (32)

Substituting (31) and (32) in (28), we obtain
\[ u(t) = -\ddot{k}(0)f. \]  \hfill (33)

Integrating (26) by considering (33), we obtain
\[ \eta_1(0) = -\ddot{k}(0)f^2T. \]  \hfill (34)

Considering (31) and (34), the initial conditions in (29) and (30) will become
\[ \ddot{x}(0) = \dot{x}_0 - \theta p_{11}(0)f + \theta p_{12}(0)\ddot{k}(0)f^2T \]  \hfill (35)

and
\[ \ddot{k}(0) = \dot{k}_0 - \theta p_{12}(0)f + \theta p_{22}(0)\ddot{k}(0)f^2T. \]  \hfill (36)

Integrating (29), by considering (32) and (33), we obtain
\[ \ddot{x}(T) = -\ddot{k}(0)^2fT + \ddot{x}(0). \]  \hfill (37)

Substituting (37) in (27), we obtain
\[ f - q_T[-\ddot{k}(0)^2fT + \ddot{x}(0)] = 0. \]  \hfill (38)

Evaluating (38), considering (35) and (36), we obtain that \( f \) is the solution of the following 5th order polynomial:
\[ \Phi(f) = \sum_{i=0}^{5} \alpha_i f^i, \]  \hfill (39)

where
\[ \alpha_0 = -q_T\ddot{x}_0, \]
\[ \alpha_1 = 1 + q_T[\dot{T}\ddot{k}_0 + \theta p_{11}(0)], \]
\[ \alpha_2 = \dot{T}q_T[2\ddot{k}_0\ddot{x}_0 - 3p_{12}(0)\dot{k}_0], \]
\[ \alpha_3 = 2\dot{T}[p_{22}(0)\ddot{x}_0 - \theta q_T(p_{12}^2(0)p_{11}(0))], \]
\[ \alpha_4 = (\dot{T})^2 p_{22}(0)q_T[-p_{22}(0)\ddot{x}_0 + p_{12}(0)\dot{k}_0], \]
\[ \alpha_5 = (\dot{T})^2 p_{22}(0)[-p_{22}(0) + \theta q_T(p_{12}^2(0) - p_{22}(0))]. \]

Considering (36) and (33), we conclude with
\[ u_{cl} = \varphi(\ddot{x}_0, P_0, T, t) = -\frac{[\ddot{k}_0 - \theta p_{12}(0)f]}{1 - \theta p_{22}(0)f^2T}, \]  \hfill (40)

where \( f \) is a solution of the 5th order polynomial (39):

The closed-loop control strategy will become
\[ u^* = -\frac{[\ddot{k}(t) - \theta p_{12}(t)f]}{1 - \theta p_{22}(t)f^2T_{go}}. \]  \hfill (41)

Remark 3. It is interesting to note that a more general case, where
\[ A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \]  \hfill (42)

leads to the following result, see Appendix A:
\[ u^* = -\frac{\hat{b}[\ddot{k}(t) - \theta p_{12}(t)f]}{1 - \theta p_{22}(t)f^2T_{go}} \exp[(a_2 - a_1)t], \]  \hfill (43)

where
\[ \hat{b} = \exp(-a_2f), \]  \hfill (44)
\[ \hat{T}_{go} = \exp(2a_1t) \times \frac{\exp[2(a_2 - a_1)T] - \exp[2(a_2 - a_1)t]}{2(a_2 - a_1)}, \]  \hfill (45)
\[ \hat{f} = f \exp(a_1\hat{T}_{go}) = \hat{f} \exp[a_1(\hat{T}_{go} - T)]. \]  \hfill (46)

Such a result is useful for a more general model.

7. Conclusions

We showed that in case of bilinear systems, control and estimation could not be designed separately. The estimation (which means learning the uncertainty \( k \)) at every instant is affected by the control; therefore the control performs a dual role of regulation and learning. Finding an optimal adaptive control (control which estimates) for bilinear systems is equivalent to solving a dual
control problem. We found an analytic closed-form solution in a deterministic way. This approach can be viewed as a new way of solving and understanding adaptive control.

The most interesting feature of this work is that the problem is solvable analytically. This is in contrast to stochastic optimal control approaches where only approximations are feasible. The analytic solution has a great contribution to the understanding of how an optimal dual-nonlinear controller works. From the illustrating example it follows that the cost function may have more than one local minima depending on the number of the solutions of the related (fifth order in this particular case) polynomial equation. The desired solution is associated with the minimum cost with respect to the controller. As the data force the estimates and pseudo-error variables, the desired solution can change. Thus, high control, which may be desirable for learning the parameters, can change to low control, that is desirable for keeping the low cost. Multiple minima in cost function were also mentioned in Sternby (1976) for a different approach.

To motivate the use of the dual control scheme, we show how it can be applied to set promotion rates for a new product, when little is known on the effects of promotional spending on sales rates and when they change with time. In order to determine promotional expenditures, most of the marketing researches measured the effectiveness of the promotion (such as advertising) on sales performance by using regression analysis with available data. When regression is used, it is usually applied to promotional expenditure data that were previously determined with the hope of maximizing profit or sales performance, but not to improve the estimation of the sales–promotion relationship. The proposed adaptive control scheme may offer some more powerful results in such marketing operations.

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Appendix A. Derivation of (43)–(46) (Remark 3)

Considering (42), Eqs. (14), (15) and (18) will become

\[ \dot{\eta}_1 = -a_1 \eta_1, \quad \eta_1(T) = f, \]  
(A.1)

\[ \dot{\eta}_2 = -(u \eta_1 + a_2 \eta_2), \quad \eta_2(T) = 0, \]  
(A.2)

where

\[ f = q_r \bar{x}(T), \]  
(A.3)

\[ u = -\bar{k} \eta_1 \]  
(A.4)

and

\[ \ddot{x} = a_1 \dot{x} + u \ddot{k}, \]

\[ \bar{x}(0) = \bar{x}_0 - \ddot{\theta} [p_{11}(0) \eta_1(0) + p_{12}(0) \eta_2(0)], \]  
(A.5)

\[ \ddot{k} = a_2 \ddot{k}, \]

\[ \bar{k}(0) = \bar{k}_0 - \theta [p_{21}(0) \eta_1(0) + p_{22}(0) \eta_2(0)]. \]  
(A.6)

Integrating (A.1) we obtain

\[ \eta_1(t) = f \exp[a_1(T - t)] = \bar{f} \exp[-a_1 t], \]  
(A.7)

where

\[ \bar{f} = f \exp[a_1 T] \]  
(A.8)

and

\[ \bar{k}(t) = \bar{k}(0) \exp(a_2 t). \]  
(A.9)

Substituting (A.7) and (A.9) in (A.4), we obtain

\[ u(t) = -\bar{k}(0) \bar{f} \exp[(a_2 - a_1)t]. \]  
(A.10)

Integrating (A.2) by considering (A.10), we obtain

\[ \eta_2(0) = -\bar{k}(0) \bar{f}^2 \bar{T}, \]  
(A.11)

where

\[ \bar{T} = \frac{\exp[2(a_2 - a_1)T] - 1}{2(a_2 - a_1)}. \]  
(A.12)

Considering (A.7) and (A.11), the initial conditions in (A.5) and (A.6) will become
\( \ddot{x}(0) = \dot{x}_0 - \theta p_{11}(0) \ddot{f} + \theta p_{12}(0) \dot{k}(0) \dot{f}^2 \dot{T} \)  
(A.13)

and

\( \ddot{k}(0) = \dot{k}_0 - \theta p_{12}(0) \ddot{f} + \theta p_{22}(0) \dot{k}(0) \dot{f}^2 \dot{T}. \)  
(A.14)

Integrating (A.5), by considering (A.9) and (A.10), we obtain

\[ \ddot{x}(T) = [-\ddot{k}(0)^2 \dddot{f} \dot{T} + \dddot{x}(0) \exp[a_1 T]. \]  
(A.15)

Substituting (A.15) in (A.3), we obtain

\[ \ddot{f} - \dddot{q}_T(-\dddot{k}(0)^2 \dddot{f} \dot{T} + \dddot{x}(0)) = 0, \]  
(A.16)

where

\[ \dddot{q}_T = q_T \exp[2a_1 T]. \]  
(A.17)

Evaluating (A.16), considering (A.13) and (A.14), we obtain that \( \ddot{f} \) is the solution of the following 5th order polynomial:

\[ \Phi(\dddot{f}) = \sum_{j=0}^{5} a_j \dddot{f}^j, \]  
(A.18)

where

\[ a_0 = -\dddot{q}_T \dddot{x}_0, \]

\[ a_1 = 1 + \dddot{q}_T [\dddot{k}_0 + \theta p_{11}(0)] , \]

\[ a_2 = \theta T \dddot{q}_T [2p_{22}(0) \dddot{x}_0 - 3p_{12}(0) \dddot{k}_0], \]

\[ a_3 = 2 \theta T [-(p_{22}(0) + \theta \dddot{q}_T (p_{12}^2(0)p_{11}(0))], \]

\[ a_4 = (\theta T)^2 p_{22}(0) \dddot{q}_T [-(p_{22}(0) \dddot{x}_0) + [p_{12}(0) \dddot{k}_0], \]

\[ a_5 = (\theta T)^2 p_{22}(0) [-(p_{22}(0) + \theta \dddot{q}_T (p_{12}^2(0) - p_{22}(0))]. \]

Considering (A.14) and (A.10), we conclude with

\[ u_{\text{ol}} = \varphi(\dddot{x}_0, P_0, T, t) \]

\[ = -\frac{\dddot{k}_0 - \theta p_{12}(0) \dddot{f}}{1 - \theta p_{22}(0) b^2 \dot{f}^2 \dot{T}} \exp[(a_2 - a_1) T]. \]  
(A.19)

The closed-loop control strategy will become

\[ u^* = -\frac{\dddot{b}[\dddot{k}(t) - \theta p_{12}(t) \dddot{f}]}{\exp[(a_2 - a_1) T]}. \]  
(A.20)

where

\[ \dddot{b} = \exp(-a_2 t), \]  
(A.21)

\[ \dddot{T}_{go} = \exp(2a_1 t) \times \exp[\frac{2(a_2 - a_1) T}{2(a_2 - a_1)}] \]  
(A.22)

\[ \dddot{f} = \dddot{f} \exp(a_1 \dddot{T}_{go}) \]  
(A.23)

and \( \dddot{f} \) is the solution of (A.18).

References

Deissenberg, Ch., Stoppler, S., 1983. Optimal information gathering and planning policies of profit-maximizing firm. Policy and Information 7, 49–75.