Quantum Suppression of Diffusion on Stochastic Webs

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(Received 24 February 1994)

Quantum suppression of diffusion on stochastic webs is shown to take place for the kicked harmonic oscillator in the form of exactly *periodic* recurrences. This phenomenon occurs, in general, only if three conditions are satisfied: (1) The kicking potential is *odd*, up to an additive constant. (2) The web is *crystalline* with *square* or *hexagonal* symmetry. (3) A dimensionless $\hbar$ assumes *integer* values. The nature of the phenomenon and its sensitivity to small perturbations are examined in terms of generalized kicked Harper models and the theory of topological Chern invariants.

PACS numbers: 05.45.+b

During the last decade, the problem of chaos in classical Hamiltonian systems and in their quantum counterparts (problem of "quantum chaos") has been studied extensively in the case of 1D periodically kicked systems described by the general Hamiltonian

$$H = H_0 + KV(x) \sum_{s=-\infty}^{\infty} \delta(t - sT) . \qquad (1)$$

Here H_0 is some time-independent Hamiltonian, K is a parameter, $V(x)$ is a periodic function of x, and T is the period. A well-known class of such systems are the kicked rotors (KR) with $H_0 = p^2/2$, where p has the meaning of angular momentum (x is an angle variable). The corresponding classical Poincaré maps are the well-known standard maps, exhibiting the generic mixture of regular and chaotic motions on all scales of phase space [1]. In general, by increasing the parameter K, bounded chaos turns into global chaos [2], featuring unbounded diffusion in the p direction [1,3]. In the quantum case, this diffusion is generically "suppressed": The expectation value of p^2 in any wave packet is bounded, exhibiting quasiperiodic recurrences [4], due to a pure-point quasienergy spectrum [i.e., the spectrum of the one-period evolution operator for (1)]. The occurrence of such a spectrum seems to be quite generic on the basis of the well-known equivalence of the quantum dynamics to a 1D Anderson model [5].

A second important class of model systems (1) are the kicked harmonic oscillators (KHOs), with $H_0 = p^2/2 + \omega^2 x^2/2$. These systems emerge in the classic problem of a charged particle interacting with an electromagnetic wave in a transverse magnetic field [6], and seem to exhibit properties qualitatively different in nature from those of the KR. Since the harmonic oscillator is a degenerate system (linear in the action), the nonlinear perturbation in (1) is strong (in the sense of KAM theory) for all values of K, especially under resonance conditions $\omega T = 2\pi m/n$ (m and n are relatively prime integers). One then expects, on the basis of general arguments [6,7], that unbounded chaotic motion should exist for arbitrarily small values of K in the resonance case. This motion takes place on a "stochastic web" [8], and exhibits a diffusive behavior [9–11]. For $n = 3, 4, 6$, the web has crystalline symmetry (triangular, square, hexagonal), see Fig. 1, while for all other values of $n > 4$ it has quasicrystalline symmetry.

The problem of quantum chaos on stochastic webs has been studied extensively in recent years [12–21], but mostly in the crystalline case $m/n = 1/4$ and in the framework of the so-called kicked Harper (KH) model. The symmetric KH Hamiltonian [6], for general periodic $V(x)$ $[V(x + 2\pi) = V(x)]$, is given by

$$H_{\mathrm{KH}} = \frac{K}{2T} V(v) + \frac{K}{2T} V(u) \sum_{s=-\infty}^{\infty} \delta(t/4T - s) , \qquad (2)$$

where $u = p/\omega$ and $v = -x$. Originally [6], the Hamiltonian (2), with $V(x) = -\cos x$, was introduced as a first-order approximation in K ($K \ll 1$) of the classical map M^4, where M is the Poincaré map of the KHO. One can show, however, that the KH model (2) is *exactly* related to the KHO for general *even* $V(x)$ [see Eq. (13) below and Refs. [21,22]]. KH models have been extensively studied in their more general, nonsymmetric version, with $KV(v)$ in (2) replaced by $LV(v)$, $L \neq K$ [23]. For rational values of $\rho \equiv \hbar/2\pi\omega$, the quasienergy spectrum of a

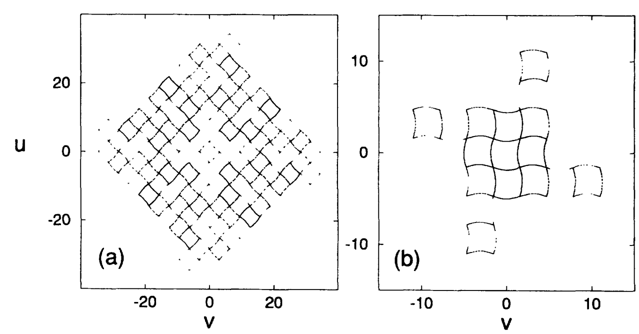

FIG. 1. Portion of the stochastic web for $m/n = 1/4$ and $\kappa = K/\omega = 0.6$, generated by 40 000 iterations of the Poincaré map with the potential: (a) $V(x) = -\cos x$, (b) $V(x) = \sin x$. Notice that the diffusion in case (b) is slower than in case (a). In fact, the residue (measure of instability [2]) at the hyperbolic points is given in the two cases by (a) $R = -\kappa^2 - \kappa^4/4$, (b) $R = -\kappa^4/4$. At large values of κ, the diffusion rates in the two cases are almost the same.

0031-9007/94/73(12)/1609(4)$06.00

general KH model is absolutely continuous, exhibiting a band structure, and $\langle p^2\rangle$ evolves ballistically ($\langle p^2\rangle \propto t^2$), as in the quantum resonance case of the KR [24]. However, in the generic case of irrational ρ, there are significant differences between the quantum dynamics of KH and KR systems. It has been proven recently [20] that KH models generically have a continuous spectrum, independently of the value of ρ. This spectrum is a consequence of the phase-space translational symmetry of these models (and also of the "crystalline" KHOs) [17,19,20], leading to a conserved quasimomentum η even for irrational ρ. As indicated by numerical calculations [12,15] for $V(x) = -\cos(x)$, the spectrum of some nonsymmetric KH models at fixed η may be pure point, and the quantum motion is then localized as in the KR case. For the symmetric KH model, however, one always observes a completely delocalized motion [12–16,20], indicating a continuous (apparently singular continuous) spectrum also at fixed η. This delocalization manifests itself in a slow decay of the autocorrelation function [16] and in an unbounded quantum diffusion of $\langle p^2\rangle$ [12–15]. This diffusion may be normal or anomalous, reflecting fractal properties of the quasienergy spectrum [14–16]. A quantum localized motion was numerically observed in the quasicrystalline case $m/n = 1/5$ of the KHO [21], but it was not related to properties of the quasienergy spectrum and eigenstates, as in the KR case. Thus, the possibility of quantum suppression of diffusion on the symmetric stochastic webs of the KHO, due to a well-defined mechanism, appears to be still unclear.

In this Letter, we show that this quantum suppression of diffusion can indeed take place in a very simple way: exactly *periodic* recurrences with the natural time period nT (i.e., $\langle\hat{O}\rangle$ is periodic with this period for an arbitrary operator $\hat{O}$). We show that this phenomenon occurs, in general, only if three conditions are satisfied: (1) The potential $V(x)$ is *odd*, up to an additive constant; we are unaware of a consideration of such potentials in the literature, but see an example in Fig. 1(b). (2) The web is *crystalline* with *square* ($n = 4$) or *hexagonal* ($n = 6$) symmetry [the triangular ($n = 3$) and quasicrystalline symmetries are thus excluded]. (3) ρ is integer for $n = 4$, while $\sqrt{3}\rho/2$ is integer for $n = 6$. As such, this phenomenon is quite different from the one in the KR case (occurring for irrational values of $\hbar T/2\pi$, and characterized by quasiperiodic recurrences) [4,5]. It is, however, similar in some aspects to the one noticed recently [25] in the two-sided kicked rotor (see discussion below). The nature of the phenomenon and its sensitivity to small perturbations will be examined in terms of generalized KH models (2) and the theory of topological Chern invariants [18,22,26].

To prove the statements above, let $V(x)$ be a general periodic function with period 2π. We define $\gamma = \omega T$, $z = (2\pi\rho)^{1/2} = (\hbar/\omega)^{1/2}$, $\alpha = iz/\sqrt{2}$, and the annihilation operator $a = (v - iu)/\sqrt{2}z$. The general evolution operator U for the KHO, from time -0 to time $T - 0$, may then be written as follows:

$$U = U_\gamma U_K(\alpha) \equiv \exp\left[-i\gamma\left(a^\dagger a + \frac{1}{2}\right)\right]\exp\left[-i\frac{K}{\hbar}\hat{V}(\alpha)\right]. \quad (3)$$

Here

$$\hat{V}(\alpha) = \sum_{l=-\infty}^{\infty} V_l D(l\alpha), \quad (4)$$

V_l are the Fourier coefficients of $V(x)$, and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is an element of the Heisenberg-Weil group of phase-space translations [27]. Now, the evolution operator for the free harmonic oscillator, given by U_γ in (3), acts on an arbitrary (polynomial) function of a and $a^\dagger$ as a rotation operator in phase space [27]:

$$U_\gamma f(a^\dagger, a)U_\gamma^{-1} = f(a^\dagger e^{-i\gamma}, ae^{i\gamma}). \quad (5)$$

Using Eq. (5) in (3), we easily obtain

$$U^s = \left\{\prod_{j=1}^{s}\exp\left[-i\frac{K}{\hbar}\hat{V}(\alpha_j)\right]\right\}U_{s\gamma}, \quad (6)$$

where $\alpha_j \equiv \alpha e^{-ij\gamma}$, and the factors under the product sign are arranged from left to right in order of increasing j. In the resonance case $\gamma = 2\pi m/n$, one has, identically, $U_{n\gamma} = U_\gamma^n = (-1)^m$. We then get from (6)

$$U^n = (-1)^m\prod_{j=1}^{n}\exp\left[-i\frac{K}{\hbar}\hat{V}(\alpha_j)\right]. \quad (7)$$

The operator (7) involves a regular n-pointed "star," consisting of the n vectors α_j in the complex plane. This is completely analogous to the classical case [6,7].

Now, the requirement of exactly periodic recurrences with the period nT is equivalent to requiring that the operator U^n in (7) is a constant phase factor. Given two arbitrary operators A and B, one has the formula [28]

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \cdots\right). \quad (8)$$

By applying (8) $n - 1$ times to the product of exponentials in (7), we get

$$U^n = (-1)^m\exp\left[-i\frac{K}{\hbar}\sum_{j=1}^{n}\hat{V}(\alpha_j) + C\right], \quad (9)$$

where C is an infinite expansion involving only repeated commutators of the operators $\hat{V}(\alpha_j)$. These repeated commutators are polynomials of order ≥ 2 in the Fourier coefficients V_l [see (4)], while the sum in (9) is a linear combination of the V_l's. It is then clear that for U^n to be a constant phase factor for general $V(x)$ in some class of potentials, one must have, identically,

$$\sum_{j=1}^{n} \hat{V}(\alpha_j) = \beta, \quad [\hat{V}(\alpha_j), \hat{V}(\alpha_{j'})] = \beta_{j,j'}, \qquad (10)$$

for all j, $j' = 1, \ldots, n$, where β and $\beta_{j,j'}$ are constants. Using (4) with $D(l\alpha)$ written in the normal form [27], and expanding in powers of $a^\dagger$ and a, we get the normally ordered expression

$$\frac{1}{n}\sum_{j=1}^{n} \hat{V}(\alpha_j) = V_0 + \sum_{l=1}^{\infty} e^{-l^2 z^2/4} \sum_{r=0}^{\infty} \sum_{s=-[r/n]}^{\infty} \left(\frac{ilz}{\sqrt{2}}\right)^{2r+sn} [V_l + V_{-l}(-1)^{2r+sn}] \frac{(a^\dagger)^r a^{r+sn}}{r!(r+sn)!}, \qquad (11)$$

where $[r/n]$ is the integer part of r/n. Using Eq. (11), it is easy to show that the first relation in (10) can be satisfied only if $\beta = nV_0$, n is even, and $V_{-l} = -V_l$ $(l > 0)$. The function $V(x) - V_0$ is then odd. Consider now the commutation relation [27]

$$D(\alpha')D(\alpha) = \exp[2i\,\mathrm{Im}(\alpha'\alpha^*)]D(\alpha)D(\alpha'). \qquad (12)$$

It follows from (4) and (12) that the only way the second relation in (10) can be satisfied for general (odd) $V(x) - V_0$ is that $\beta_{j,j'} = 0$, with all the n phase-space translations $D(\alpha_j)$ commuting with each other. Using (12), the requirement of exact commutation implies that $\rho \sin(j\gamma) = q_j$ for all j, where q_j is an integer. The last relation means that all the n vectors of the star are linearly dependent with integer coefficients. This is possible only for a crystalline symmetry. Since n must be even, the only possible cases are $n = 4$ or $n = 6$. For $n = 4$, $\rho = q$ (an integer), while, for $n = 6$, $\sqrt{3}\rho/2 = q$. This completes the proof.

We now examine in some detail the nature of this phenomenon and its sensitivity to small perturbations of ρ and $V(x)$. For simplicity, we consider here only the case $n = 4$. The operator (7), for general values of ρ and for $V(x)$ of well-defined parity, may be written in this case as follows

$$U^4 = \begin{cases} -\tilde{U}_{\mathrm{KH}}^2(K/2), & V(x)\ \text{even}, \\ -U_{\mathrm{KH}}^{-1}(K/2)\tilde{U}_{\mathrm{KH}}(K/2), & V(x)\ \text{odd}. \end{cases} \qquad (13)$$

Here $\tilde{U}_{\mathrm{KH}}(K) = SU_{\mathrm{KH}}(K)S^{-1}$, $S = \exp[iKV(v)/\hbar]$, and $U_{\mathrm{KH}}(K)$ is the evolution operator for the generalized KH model (2) in one period (from $t = -0$ to $t = 4T - 0$):

$$U_{\mathrm{KH}}(K) = \exp\left[-\frac{2iK}{\hbar}V(v)\right]\exp\left[-\frac{2iK}{\hbar}V(u)\right]. \qquad (14)$$

For $V(x)$ even, Eq. (13) means that the KHO is exactly described, up to the unitary transformation S, by the KH model at parameter $K/2$ (see also Refs. [21,22]). In particular, for ρ integer, all the operators in (13) and (14) commute with each other, so that $U^4 = -U_{\mathrm{KH}}(K)$. Integer values of ρ correspond to the fundamental quantum resonances of the KH model [and thus also of the KHO for $V(x)$ even], with $\langle p^2 \rangle \propto t^2$ as in the KR case [24]. On the other hand, for odd $V(x)$ and ρ integer, the equality $\tilde{U}_{\mathrm{KH}} = U_{\mathrm{KH}}$ leads to $U^4 = -1$. Thus, the quantum suppression of diffusion may be viewed as a phenomenon of quantum "antiresonance" [29], resulting from a "cancellation" of the quantum resonances of the two KH models U_{KH}^{-1} and $\tilde{U}_{\mathrm{KH}}$. The details of this cancellation effect will be given elsewhere [22]. Here we only mention that for ρ close enough to some integer q, $\rho = q + \epsilon$, $U^4 \approx -\exp[-(K/2\pi\omega q)^2 C]$, where $C = [V(v), V(u)] = O(\epsilon)$. The quasienergy spectrum has thus a width $\approx O(\epsilon)$, and degenerates to the value -1 as $\epsilon \to 0$. In this limit one finds that $\langle p^2 \rangle$ grows either diffusively or ballistically at a rate not larger than $O(\epsilon)$.

Let us now consider perturbations in $V(x)$. For ρ integer and general $V(x)$, one obtains from (13) that $U^4 = -\exp\{-2iK[V_e(u) + V_e(v)]/\hbar\}$, where $V_e(x) = [V(x) + V(-x)]/2$ is the even part of $V(x)$. Here $V_e(x)$ is viewed as a perturbation δV. The quasienergy eigenstates of U^4 are independent of $V(x)$ and are given by [22]

$$\psi_{\mathbf{w}}(v) = \sum_{l=-\infty}^{\infty} \exp(-ilw_2/\rho)\delta(v - w_1 + 2\pi l), \qquad (15)$$

where the "quasimomentum" $\mathbf{w} \equiv (w_1, w_2)$ ranges in the "Brillouin" zone: $0 \le w_1 < 2\pi$, $0 \le w_2 < 2\pi\rho$. The corresponding eigenvalues are $-\exp[-iE(\mathbf{w})/\hbar]$, where the quasienergies $E(\mathbf{w})$ span precisely one band, $E(\mathbf{w}) = 2K[V_e(w_1) + V_e(w_2)]$. An interesting example is $V(x) = \sin(x + \delta x)$. Here δx corresponds to a perturbation of the conserved y component of the momentum in the original problem of the periodically kicked charge in a magnetic field [6]. In this case, the bandwidth is proportional to $\sin(\delta x)$, and the asymptotic growth of $\langle p^2 \rangle$ is proportional to $[\sin(\delta x)t]^2$. Thus, for both kinds of perturbations, the quantum suppression of diffusion is still approximately observed on time intervals proportional, at least, to ϵ^{-1} or $(\delta V)^{-1}$.

It is interesting to consider the case of rational values of ρ close to an integer q, $\rho = q + 1/N$ (N is a large integer). In this case, and for general $V(x)$, the quasienergy spectrum consists precisely of N bands [22]. As in Ref. [18] one can associate with each band a Chern integer σ ("quantum Hall conductance" [26]), defined by a topologically invariant integral over the Brillouin zone. In Ref. [18], only the case $q = 0$ was considered for KH models in the semiclassical limit of large N. It was observed that eigenstates spread over the chaotic region (featuring diffusive mobility) are usually characterized by $\sigma \neq 0$, while eigenstates localized, e.g., on regular regions are characterized by $\sigma = 0$. In our case, however, $q > 0$, and one can show [22] that the value $\sigma = 0$ can *never* arise, even for odd $V(x)$. Thus, for example, the single band of eigenstates (15) (for $\rho = q$) is associated with $\sigma = 1$ [22,26]. This fact, of course, does not contradict the semiclassical results of Ref. [18], since the quantum periodic recurrences for $\rho = q$ have no classical analog. A quantum-dynamical interpretation of the Chern integers $\sigma \neq 0$ near $\rho = q > 0$ is an open question.

In conclusion, we have shown that quantum suppression of diffusion on symmetric stochastic webs is possible in the form of exactly periodic recurrences. This phenomenon can occur only for odd potentials (up to an additive constant) and on crystalline webs with square or hexagonal symmetry. While the quasienergy spectra for such webs are expected to be, in general, continuous, the phenomenon emerges as a quantum "antiresonance" effect at integer values of a dimensionless $\hbar$. At these values of $\hbar$, the quasienergy spectrum reduces to a single infinitely degenerate level. The phenomenon persists under perturbations on time intervals proportional, at least, to the inverse perturbation strength. Exactly periodic recurrences occur also in the two-sided kicked rotor [described by (1) with $H_0 = p^2/2$ and $\delta(t - sT)$ replaced by $\delta(t - sT) - \delta(t - (s + 1/2)T)$] [25], for suitable values of the parameters. This phenomenon occurs for general $V(x)$ (not necessarily odd), and may also be viewed as a quantum antiresonance effect, but in KR spectra (instead of KH spectra). The phenomenon appears then to be much more stable under perturbations than in our case [29]. However, while quantum recurrences (periodic or quasiperiodic) are quite generic in KR systems, the periodic recurrences in the KHO are apparently the only known case of exact quantum suppression of diffusion on symmetric stochastic webs.

The author is indebted to D. Shepelyansky for stimulating discussions and helpful comments. Useful discussions with E. Eisenberg, N. Shnerb, F. Borgonovi, I. Guarneri, G. M. Zaslavsky, S. Fishman, M. Feingold, and J. Levitan are gratefully acknowledged.

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