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Composite von Neumann lattice

I. Dana
Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel
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A new general subset of coherent states, which is complete and overcomplete by just one element, is introduced. It is the union of N arbitrary nonintersecting von Neumann lattices whose unit-cell areas have a harmonic mean of Nh. First applications are given in the problems of crystal electrons in rational magnetic fields, incommensurate structures in a magnetic field, and quantum dynamics.

Coherent states[1,2] have a number of remarkable properties which make them useful in many fields of quantum physics.[2–4] These states are minimal wave packets usually defined as eigenstates of the annihilation operator $\hat{a}=(1/\sqrt{2}\lambda)[\hat{x}+(i/\hbar)\lambda^2\hat{p}]$, where λ is associated with the constants m and ω of a harmonic oscillator, $\lambda^2=\hbar/m\omega$. The eigenvalues α of $\hat{a}$ fill the whole complex plane and can be written as $\alpha=(1/\sqrt{2}\lambda)[\bar{x}+(i/\hbar)\lambda^2\bar{p}]$, where $\bar{x}$ and $\bar{p}$ are the coordinate and momentum expectation values in the coherent state $|\alpha\rangle$. The correspondence $|\alpha\rangle \leftrightarrow (\bar{x},\bar{p})$ defines then the classical description of coherent states.[5]

A quantum-mechanical representation based on the full set of coherent states[2,6] has unusual features which follow from the fact that this set is nonorthogonal and overcomplete. The nonorthogonality is not surprising since the coherent states $|\alpha\rangle$ are eigenstates of the non-Hermitian operator $\hat{a}$. The overcompleteness is connected with the nonorthogonality and means that the set of coherent states contains subsets which are complete. By completeness of a subset we intend the following: any square-integrable state which is orthogonal to all the states of the subset is the zero state. A physical way of how to choose such a subset was already suggested by von Neumann[7] in connection with the most exact simultaneous measurements of the coordinate and momentum in quantum mechanics. The von Neumann set consists of the states $|\alpha_{mn}\rangle$, where $\alpha_{mn}=m\omega_1+n\omega_2$ (m and n are all integers) form a lattice in the complex plane with a unit-cell area $S\equiv\mathrm{Im}(\omega_1^*\omega_2)=\pi$. In the $\bar{x}$-$\bar{p}$ phase plane this corresponds to a lattice with a unit-cell area of h, the Planck constant. The completeness of the von Neumann lattice of states was proved later in several ways[8–10] and it was shown[8,11] that this set is overcomplete by just one element (namely, it remains complete if any one but only one element is removed). Besides its conceptual importance, the von Neumann lattice is useful as an expansion basis in quantum mechanics,[12] and has several applications.[13–15]

The von Neumann lattice is apparently the only known subset of coherent states which is exactly complete (up to one element). In fact, other complete subsets of coherent states, known as Bargmann characteristic sets,[6,8] are generally overcomplete by any finite number of elements. It is therefore natural to ask whether other subsets of coherent states exist having the same completeness properties as the von Neumann lattice. Such a general set is introduced in this paper. Consider the set $\{|\alpha_{mn,j}\rangle\}$ where

$$\alpha_{mn,j}=m\omega_{1j}+n\omega_{2j}+\alpha_j \quad . \tag{1}$$

Here m,n are all integers, $j=1,\ldots,N$, ω_{1j} and ω_{2j} are $2N$ arbitrary complex numbers with $\mathrm{Im}(\omega_{1j}^*\omega_{2j})\neq 0$, α_j are N arbitrary complex numbers, and no two values from (1) coincide for different triples (mn,j). The set $\{|\alpha_{mn,j}\rangle\}$ is simply the union of N arbitrary nonintersecting von Neumann lattices with unit-cell areas $S_j=\mathrm{Im}(\omega_{1j}^*\omega_{2j})$. Here the term "von Neumann lattice" is used also in the general case $S\neq\pi$. For $S>\pi$ a von Neumann lattice is incomplete.[8,9] Sets of superimposed incomplete von Neumann lattices appear in a natural way in physical problems of recent interest (see the discussion later). The completeness of such sets has not yet been investigated. Here we give a rigorous proof of the following statements. The set $\{|\alpha_{mn,j}\rangle\}$ is complete and overcomplete by just one element if and only if the harmonic mean of S_j, $j=1,\ldots,N$, is equal to $N\pi$ (or Nh in the $\bar{x}$-$\bar{p}$ phase plane):

$$\bar{S}\equiv N\left(\sum_{j=1}^{N}S_j^{-1}\right)^{-1}=N\pi \quad . \tag{2}$$

The set $\{|\alpha_{mn,j}\rangle\}$ is overcomplete by any finite number of elements for $\bar{S}<N\pi$; and, finally, it is not complete for $\bar{S}>N\pi$. This is in full analogy with the simple von Neumann lattice ($N=1$).[8] We shall call the set $\{|\alpha_{mn,j}\rangle\}$ "composite von Neumann lattice." However irregular the distribution of points (1) in the complex plane may be, the condition (2) assures that there is, on the average, one point in a Planck area of h. This seems to be the general condition for the exact completeness (up to one element) of a set of coherent states.

The proof is as follows: by definition, the set $\{|\alpha_{mn,j}\rangle\}$ is complete if and only if from $\langle\psi'|\alpha_{mn,j}\rangle=0$, for all $\alpha_{mn,j}$; it follows that either $\langle\psi'|\psi'\rangle=0$ or $\langle\psi'|\psi'\rangle=\infty$. For a square-integrable state $|\psi\rangle$, the amplitude $\langle\psi|\alpha\rangle$ can be written in the well-known form[2,8]: $\langle\psi|\alpha\rangle=\exp(|\alpha|^2/2)\psi(\alpha)$, where $\psi(\alpha)$ is an entire analytic function of the complex variable α. The norm $\langle\psi|\psi\rangle$ can be expressed as follows[2,8]:

$$\langle\psi|\psi\rangle=\frac{1}{\pi}\int d^2\alpha\exp(-|\alpha|^2)|\psi(\alpha)|^2 \quad . \tag{3}$$

One can establish a one-to-one correspondence between the square-integrable states $|\psi\rangle$ and the entire functions $\psi(\alpha)$ for which the integral in (3) is finite.[2,8] In particular, a square-integrable state which is orthogonal to all the states $|\alpha_{mn,j}\rangle$ will correspond to an entire function $\psi(\alpha)$ having zeros at the points $\alpha_{mn,j}$. If this function is not the zero function the set $\{|\alpha_{mn,j}\rangle\}$ is not complete. If, on the other hand, for any entire function $\psi(\alpha)$ with zeros at the points $\alpha_{mn,j}$ the integral in (3) diverges, the set $\{|\alpha_{mn,j}\rangle\}$ is complete. The integral in (3) is finite if and only if one has, asymptotically, $|\psi(\alpha)| < \exp(|\alpha|^2/2)$. This inequality can be satisfied if and only if the order of growth of $\psi(\alpha)$ is smaller than 2 or if it is equal to 2 and the type of $\psi(\alpha)$ is smaller than $\frac{1}{2}$. We recall here that the order of growth β and the type γ of an entire function[16] $\psi(\alpha)$ are the lower bounds of positive numbers k and A, respectively, such that asymptotically $|\psi(\alpha)| < \exp(A|\alpha|^k)$. For an entire function with zeros at the points $\alpha_{mn,j}$ the order of growth β cannot be less than the exponent of convergence λ_1 of the sequence $\alpha_{mn,j}$,[16] where, by definition (the prime excludes $\alpha_{mn,j}=0$),

$$\sum_{mn,j}' |\alpha_{mn,j}|^{-\lambda_1-\epsilon} < \infty, \quad \sum_{mn,j}' |\alpha_{mn,j}|^{-\lambda_1+\delta} = \infty \quad ,$$

for arbitrarily small $\epsilon > 0$ and $\delta > 0$. Using (1) one readily finds that $\lambda_1 = 2$, so that $\beta \geqslant 2$. We shall now construct an entire function having simple zeros at the points $\alpha_{mn,j}$, order of growth $\beta = 2$, and the minimal possible type γ consistent with the given zeros.

Let us first construct an entire function having simple zeros at the lattice sites $\alpha_{mn}^{(j)} = m\omega_{1j} + n\omega_{2j}$, for some j. This is the Weierstrass σ function[17]

$$\sigma_j(\alpha) = \alpha \prod_{mn}{}' \left[1 - \frac{\alpha}{\alpha_{mn}^{(j)}}\right] \exp\left[\frac{\alpha}{\alpha_{mn}^{(j)}} + \frac{\alpha^2}{2\alpha_{mn}^{(j)2}}\right] \quad ,$$

where the prime excludes $\alpha_{00}^{(j)} = 0$. The function $\sigma_j(\alpha)$ has the following property[8]

$$|\sigma_j(\alpha)|^2 = \rho_j(\alpha, \alpha^*) \times \exp(2\mu_j|\alpha|^2 + \nu_j\alpha^{*2} + \nu_j^*\alpha^2) \quad , \tag{4}$$

where $\rho_j(\alpha, \alpha^*)$ is doubly periodic,

$$\rho_j(\alpha + \alpha_{mn}^{(j)}, \alpha^* + \alpha_{mn}^{(j)*}) = \rho_j(\alpha, \alpha^*) \quad ,$$

$\mu_j = \pi/(2S_j)$, and ν_j is a constant depending on the unit-cell dimensions.[17]

We now introduce the function

$$\bar{\psi}(\alpha) = \prod_{j=1}^{N} \exp[2\mu_j\alpha_j^*\alpha - \mu_j|\alpha_j|^2 - \nu_j^*(\alpha - \alpha_j)^2] \times \sigma_j(\alpha - \alpha_j) \quad . \tag{5}$$

The function (5) is entire and assumes simple zeros precisely at the points $\alpha_{mn,j}$. Using (4) an elementary calculation gives

$$|\bar{\psi}(\alpha)|^2 = \exp\left[\frac{N\pi}{\bar{S}}|\alpha|^2\right]\bar{\rho}(\alpha, \alpha^*) \quad , \tag{6}$$

where $\bar{S}$ is defined by (2) and

$$\bar{\rho}(\alpha, \alpha^*) = \prod_{j=1}^{N} \rho_j(\alpha - \alpha_j, \alpha^* - \alpha_j^*) \quad . \tag{7}$$

The function (7) is nonnegative, assumes zeros at the points $\alpha_{mn,j}$, and is generally almost-periodic and bounded. Relation (6) then clearly shows that the entire function $\bar{\psi}(\alpha)$ is of second order of growth and of minimal possible type $\gamma = N\pi/(2\bar{S})$ for the given zeros. To check completeness it is therefore sufficient to consider the function $\bar{\psi}(\alpha)$.

Using (6) we see that the integral in (3) for the function $\bar{\psi}(\alpha)$ converges for $\bar{S} > N\pi$, so that the set $\{|\alpha_{mn,j}\rangle\}$ is not complete in this case. For $\bar{S} \leqslant N\pi$ the integral diverges and the set $\{|\alpha_{mn,j}\rangle\}$ is complete. If any state, say $|\alpha_{m_0n_0,j_0}\rangle$, is removed from the set one has to consider the function $\bar{\psi}(\alpha)/(\alpha - \alpha_{m_0n_0,j_0})$ instead of $\bar{\psi}(\alpha)$. Then, by using the above-mentioned properties of the function $\bar{\rho}(\alpha, \alpha^*)$ and arguments quite similar to those used for the von Neumann lattice,[8] one may easily prove that the set $\{|\alpha_{mn,j}\rangle\}$ is overcomplete by any finite number of elements for $\bar{S} < N\pi$, while it is overcomplete by just one element for $\bar{S} = N\pi$. This completes the proof.

We now discuss physical problems which are characterized by a composite von Neumann lattice. As a first example we consider the problem of an electron in a two-dimensional crystal perpendicular to a uniform magnetic field $\vec{H}$. The energy spectrum generally consists of magnetic bands,[15,18,19] which result from Landau level broadening. A localized basis suitable for calculating the magnetic band structure is the so-called Pippard network of Dingle functions.[18] This is the set of functions obtained by operating on a Dingle function (corresponding to the classical circular trajectory in a magnetic field) with magnetic translations[20] in a lattice $\vec{b}_n = n_1\vec{b}_1 + n_2\vec{b}_2$, where the unit cell $\vec{b}_1 - \vec{b}_2$ encloses exactly one quantum of flux: $\vec{b}_1 \cdot \vec{b}_2 H = hc/e$. It is known[14] that a Pippard network of Dingle functions is completely analogous to a von Neumann lattice of coherent states. In particular, it follows from the completeness of the von Neumann lattice that a Pippard network spans the space of a Landau level. One usually assumes that $\vec{b}_n$ can be chosen as a superlattice of the crystal lattice $\vec{a}_n = n_1\vec{a}_1 + n_2\vec{a}_2$ (Ref. 18): $\vec{b}_1 = s_1\vec{a}_1$, and $\vec{b}_2 = s_2\vec{a}_2$, where s_1 and s_2 are integers. This corresponds to the rationality condition $s\vec{a}_1 \cdot \vec{a}_2 H = hc/e$, where $s = |s_1s_2|$. Let us consider now the more general condition[20] $s\vec{a}_1 \cdot \vec{a}_2 H = Nhc/e$, where s and N are relatively prime integers, and we assume that $s > N$ (this is always the case for accessible field strengths). The Pippard lattice $\vec{b}_n$ cannot be chosen now as a superlattice of the crystal lattice $\vec{a}_n$. The set of Dingle functions associated with the superlattice $\vec{b}_n' = n_1s_1\vec{a}_1 + n_2s_2\vec{a}_2$ $(s_1s_2 = s)$ does not span the whole space of a Landau level, since the unit cell $s_1\vec{a}_1 - s_2\vec{a}_2$ encloses now N flux quanta. This corresponds to an incomplete von Neumann lattice whose unit-cell area is Nh. However, by invoking the completeness of a set of von Neumann lattices, we see that the set of Dingle functions associated with the N superlattices $\vec{b}_{n,j}' = \vec{b}_n' + \vec{a}_j$ ($\vec{a}_j$, $j = 1, \ldots, N$, are N different crystal-lattice sites inside the unit cell $s_1\vec{a}_1 - s_2\vec{a}_2$) spans the space of a Landau level.

As a second example we consider the problem of two-dimensional incommensurate structures in a magnetic field $\vec{H}$.[21] Let $\vec{a}_n = n_1\vec{a}_1 + n_2\vec{a}_2$ and $\vec{a}_n' = g\vec{a}_n$, with g an irrational number, be the lattices of two such structures. We ask about the possibility of associating with lattice sites of both structures a complete set of Dingle functions for a Landau level. A condition for this can be easily written by using the analogy between Pippard networks and von Neumann lattices,[14] and the completeness relation (2):

$$(s^{-1} + s'^{-1}g^{-2})^{-1}\vec{a}_1 \cdot \vec{a}_2 H = hc/e \quad ,$$

where s and s' are positive integers. This condition is analogous to the rationality condition $s\vec{a}_1 \cdot \vec{a}_2 H = hc/e$ for a perfect crystal, and implies that the set of Dingle functions associated with the superlattices

$$\vec{b}_n = n_1 s_1 \vec{a}_1 + n_2 s_2 \vec{a}_2 + \vec{a}$$

and

$$\vec{b}'_n = n_1 s'_1 g \vec{a}_1 + n_2 s'_2 g \vec{a}_2 + g \vec{a}'$$

($s_1 s_2 = s$, $s'_1 s'_2 = s'$, and $\vec{a}$ and $\vec{a}'$ are two arbitrary points in the lattice $\vec{a}_n$) form a complete set of functions for a Landau level. This set is generally overcomplete by one element, except for the case in which $\vec{a} = \vec{a}' = 0$. In this case the set is exactly complete.

A third example, which we consider here only in general terms (details will be given elsewhere), is connected with the problem of quasienergy states.[3,22,23] These are solutions of the Schrödinger equation with a time-periodic Hamiltonian, and have the general form: $|\psi_w, t\rangle = e^{-iwt}|u_w, t\rangle$, where w is the quasienergy and $|u_w, t\rangle$ is time periodic with the period of the Hamiltonian. Consider the Hamiltonian

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} - P\hat{x} \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad , \tag{8}$$

describing a harmonic oscillator periodically kicked by a fixed impulse P. For rational rotation numbers, $\omega T/2\pi = p/q$ (p and q are relatively prime integers), the quasienergies can be shown to be infinitely degenerate.[24] To account for this degeneracy we notice first that under a Hamiltonian like (8) the time evolution of a coherent state follows the classical phase-plane trajectory.[3] One can now find[24] a set of points $\{\alpha_{mn,j}\}$ which can be expressed as the union of either q noncoincident congruent lattices $\alpha_{mn,j}$ ($j = 1, \ldots, q$) or disjoint subsets $\{\alpha_i\}_r$ which are invariant under the time translation $t \rightarrow t + T$. A set of quasienergy states can be associated with each invariant subset[22] $\{\alpha_i\}_r$ by forming symmetry-adapted sums of coherent states $|\alpha_i\rangle_r$. The index r of the invariant subset is then the degeneracy index, and the infinite degeneracy is completely accounted for by simply choosing the unit-cell area of each lattice as $S = q\pi$.

In summary, we have shown the existence of new general subsets of coherent states having the same completeness properties as the von Neumann lattice. These sets correspond, in general, to irregular distributions of points in the phase plane, with the average of one point in a Planck area of h. As such, these new complete sets of states may turn out to be useful in several physical applications.

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[1] E. Schrödinger, Naturwissenschaften 14, 664 (1926).
[2] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[3] A. M. Perelomov, Usp. Fiz. Nauk 123, 23 (1977) [Sov. Phys. Usp. 20, 703 (1977)], and references therein.
[4] J. Zak, Phys. Rev. B 21, 3345 (1980); G. P. Berman and G. M. Zaslavsky, Physica A 111, 17 (1982).
[5] I. R. Senitzky, Phys. Rev. Lett. 47, 1503 (1981).
[6] V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961); 20, 1 (1967).
[7] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1955), Chap. V, Sec. 4. German edition published in 1932.
[8] A. M. Perelomov, Teor. Mat. Fiz. 6, 213 (1971) [Theor. Math. Phys. 6, 156 (1971)].
[9] V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder, Rep. Math. Phys. 2, 221 (1971).
[10] H. Bacry, A. Grossmann, and J. Zak, Phys. Rev. B 12, 1118 (1975).
[11] M. Boon and J. Zak, J. Math. Phys. 22, 1090 (1981).
[12] M. Boon and J. Zak, Phys. Rev. B 18, 6744 (1978); J. Math. Phys. 19, 2308 (1978).
[13] J. Zak, Phys. Rev. B 20, 2228 (1979).
[14] M. Boon, in *Group Theoretical Methods in Physics,* edited by A. Janner, T. Janssen, annd M. Boon, Lecture Notes in Physics, Vol. 50 (Springer-Verlag, Berlin, 1976), p. 282.
[15] I. Dana and J. Zak, Phys. Rev. B 28, 811 (1983).
[16] B. Ja. Levin, *Distribution of Zeros of Entire Functions* (A.M.S., Providence, RI, 1964).
[17] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1950), p. 447.
[18] A. B. Pippard, Philos. Trans. R. Soc. London, Ser. A 256, 317 (1964).
[19] J. W. Eddy, Jr., and R. W. Stark, Phys. Rev. Lett. 48, 275 (1982).
[20] J. Zak, Phys. Rev. 134, A1602 (1964).
[21] E. Batalla, F. S. Razavi, and W. R. Datars, Phys. Rev. B 25, 2109 (1982).
[22] M. V. Berry, N. L. Balasz, M. Tabor, and V. Voros, Ann. Phys. (N.Y.) 122, 26 (1979).
[23] F. Gesztezy and H. Mitter, J. Phys. A 14, L79 (1981); S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. 49, 509 (1982); D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. 49, 833 (1982).
[24] I. Dana (unpublished).