6 February 1995

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 197 (1995) 413–416

Kicked Harper models and kicked charge in a magnetic field

Itzhack Dana

Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

Received 11 November 1994; accepted for publication 8 December 1994
Communicated by A.R. Bishop

Abstract

The symmetric kicked Harper (KH) model was originally derived as a first-order approximation (in the kicking parameter) of the problem of a kicked charge in a magnetic field (KCM). It is shown that *all* generalized KH models (symmetric or non-symmetric, periodic or not) can, in fact, be *exactly* related to KCM problems. Because of this exact relation, all the results derived recently for KH models hold straightforwardly also for the corresponding KCM problems. For the vast majority of KCM problems, however, there exist no generalized KH models related to them.

The kicked Harper (KH) model, defined by the Hamiltonian

$$H_{\mathrm{KH}} = -\frac{2L}{T}\cos(v) - \frac{2K}{T}\cos(u)\sum_{s=-\infty}^{\infty}\delta(t/T - s) \tag{1}$$

(u and v are dimensionless conjugate phase-space variables, L and K are parameters, and T is the period), has been investigated extensively in recent years [1–12] as a simple model system exhibiting a rich variety of classical and quantum dynamical properties. For some parameter values, this system features quantum-dynamical localization [5,9], i.e., suppression of classical diffusion, as in the case of the kicked rotor [13]. For other values of the parameters, it exhibits the phenomenon of quantum delocalization, i.e., quantum diffusion, which is usually anomalous [5–7,9]. This phenomenon is due to a continuous (absolutely and/or singular continuous) component in the quasienergy spectrum, exhibiting multifractal properties. In cases where there is no pure-point component in the spectrum, one also observes slow decays of the quantum autocorrelation function [8,9]. An interesting characterization of the classical–quantum correspondence in the KH dynamics was given recently [4] in terms of topological Chern integers, analogous to the quantized Hall conductances in the problem of Bloch electrons in a magnetic field [14].

A natural question is then to what extent the KH model, with such a rich dynamical behavior, actually corresponds to a realistic, physically realizable system. Originally [1,2], the symmetric ($L = K$) KH model, was derived as a first-order approximation in the parameter K ($K \ll 1$) of the problem of a kicked charge in a magnetic field (KCM). This problem, exhibiting the classical chaotic motion on stochastic webs, is described by the Hamiltonian

$$H = \tfrac{1}{2}\boldsymbol{\Pi}^2 - K\cos(x)\sum_{s=-\infty}^{\infty}\delta(t - 2\pi s/\Omega), \tag{2}$$

where $\boldsymbol{\Pi} = \boldsymbol{p} - q\boldsymbol{A}/c$ is the kinetic momentum of charge q in a (uniform) magnetic field $\boldsymbol{B}$ (with vector potential $\boldsymbol{A}$), Ω is the kicking frequency, and a unit mass is assumed (without loss of gen-

SSDI 0375-9601(94)01007-2

erality). In the case $\Omega/|\omega| = 4$ ($\omega = qB/c$ is the cyclotron frequency), the stochastic webs have square symmetry. It was shown in Ref. [1] that in this case the Poincaré map of the symmetric KH model (1), with $L = K$, $u = \Pi_x/|\omega|$, $v = \Pi_y/\omega$, and $T = 2\pi/|\omega|$, is a first-order approximation in K of the Poincaré map of (2) in the time period T, for some value of a conserved quantity (see below).

In this Letter we show that the KH model can, in fact, be *exactly* related to a KCM problem under *quite general* conditions. This exact relation holds both quantally and classically (see relations (6) and (14) below), and allows one to apply in a straightforward way all the results derived recently for KH models to the corresponding KCM problems. The relation, however, is limited to only *one* value of a conserved quantity for the KCM problem. Moreover, even such a limited relation holds only for a small sub-class of the KCM problems. Thus, the KH dynamics, while very rich, is, essentially, only a very special case of the KCM dynamics. We consider *generalized* KH models defined by the Hamiltonian

$$H_{\mathrm{KH}} = \frac{2K}{T}V_1(v) + \frac{2K}{T}V_2(u)\sum_{s=-\infty}^{\infty}\delta(t/T - s), \quad (3)$$

where $V_1(v)$ and $V_2(u)$ are *general* functions of v and u (not necessarily periodic). Correspondingly, we consider the general KCM problem described by

$$H = \tfrac{1}{2}\Pi^2 + KV(x,t)\sum_{s=-\infty}^{\infty}\delta(t - 2\pi s/\Omega), \quad (4)$$

where $V(x,t)$ is periodic in t with period $T = 2\pi/|\omega|$. We assume again that $\Omega/|\omega| = 4$, as it will become apparent later (see Eqs. (10) and (11) below) that only in this case a relation between (3) and (4) may be expected. It is crucial to write (4) in the natural coordinates of the problem, given by the conjugate pairs (x_c, y_c) (coordinates of the center of a cyclotron orbit) and (u,v). Using $\Pi_x = |\omega|u$, $\Pi_y = \omega v$ (see above), and the relation $x_c = x+\Pi_y/\omega = x+v$ (easily derivable from simple geometry), (4) is rewritten as follows,

$$H = \tfrac{1}{2}\omega^2(u^2+v^2) + KV(x_c - v, t)\sum_{s=-\infty}^{\infty}\delta(t - 2\pi s/\Omega). \quad (5)$$

Since H in (5) does not depend on y_c, x_c is *conserved* (a constant of the motion). We shall therefore treat x_c in (5) as a parameter, and use the notation $V(x_c - v, t) = W(v,t)$.

We now present our main results and their proof. Several remarks and comments will then follow. Let $U_{\mathrm{KH}}(K)$ and $U(K)$ be the quantum evolution operators for (3) and (5), respectively, from time $t = -0$ to time $t = T - 0$. The exact relation

$$U(K) = -U_{\mathrm{KH}}^{-2}(-\tfrac{1}{2}K) \quad (6)$$

holds *if and only if* the following conditions are satisfied,

$$W(v,0) = W(-v,\tfrac{1}{2}T) = V_1(v), \quad (7)$$

$$W(-u,\tfrac{1}{4}T) = W(u,\tfrac{3}{4}T) = V_2(u). \quad (8)$$

The proof is as follows. The evolution operator for (5) from time $t = 2\pi s/\Omega - 0$ to time $t = 2\pi(s+1)/\Omega - 0$ is given by

$$\begin{aligned} U_s &= \tilde{U}U_K(v, 2\pi s/\Omega) \\ &\equiv \exp\left[-\pi\mathrm{i}\omega^2(u^2+v^2)/\hbar\Omega\right] \\ &\quad\times \exp\left[-\mathrm{i}KW(v,2\pi s/\Omega)/\hbar\right]. \end{aligned} \quad (9)$$

The evolution operator of the harmonic oscillator, given by $\tilde{U}$ in (9), is essentially a rotation operator in phase space [15]. In the particular case $\Omega/|\omega| = 4$, using the commutation relation $[u,v] = \mathrm{i}\hbar/|\omega|$, we easily find that

$$\tilde{U}f(u,v)\tilde{U}^{-1} = f(-v,u),$$
$$\tilde{U}|l\rangle = \exp\left[-\tfrac{1}{2}\mathrm{i}\pi(l+\tfrac{1}{2})\right]|l\rangle, \quad (10)$$

where $f(u,v)$ is an arbitrary (polynomial) function of u and v, and $|l\rangle$ is the lth harmonic-oscillator state. By repeated application of the first equation in (10), and using the fact that $\tilde{U}^4 \equiv -1$ from the second equation, we get

$$\begin{aligned} U(K) &= U_3U_2U_1U_0 \\ &= -U_K(u,\tfrac{3}{4}T)U_K(-v,\tfrac{1}{2}T) \\ &\quad\times U_K(-u,\tfrac{1}{4}T)U_K(v,0), \end{aligned} \quad (11)$$

where all operators are defined in (9). For the KH model (3) we have

$$U_{KH}(K) = \exp\left[-2iKV_1(v)/\hbar\right]\exp\left[-2iKV_2(u)/\hbar\right]. \tag{12}$$

Using (12) to evaluate $U_{KH}^{-2}(-\frac{1}{2}K)$, and comparing the resulting expression with (11), we obtain relation (6) with the conditions (7) and (8).

Our first and main remark concerning relation (6) with (7) and (8) is as follows. Given a generalized KH model (3), one can always define a KCM problem exactly related to it by (6). This is because, given $V_1(v)$ and $V_2(u)$, one can always find a function $W(v,t)$ satisfying the conditions (7) and (8). In fact, such a function is given explicitly by

$$W(v,t) = V_1(v)f(t) + V_2(-v)f(t-\tfrac{1}{4}T) + V_1(-v)f(t-\tfrac{1}{2}T) + V_2(v)f(t-\tfrac{3}{4}T), \tag{13}$$

where $f(t)$ is any periodic function with period T satisfying $f(0) = 1$ and $f(\frac{1}{4}T) = f(\frac{1}{2}T) = f(\frac{3}{4}T) = 0$. Clearly, any other $W(v,t)$ satisfying (7) and (8) is equivalent to (13) because of the delta time periodicity in (5). Now, $W(v,t)$ is associated with some value of $x_c = x_c^{(0)}$ in (5) through $W(v,t) = V(x_c^{(0)} - v, t)$. It is then easy to check that the conditions (7) and (8) cannot be satisfied by any other function $V(x_c - v, t)$, for $x_c \neq x_c^{(0)}$, unless all the functions V, V_1 and V_2 are periodic with the same period. In this case, one can still say that the conditions are satisfied by only one value of x_c per period. It is clear, however, that *given* a KCM problem, the function $W(v,t) = V(x_c - v, t)$ does not assume in general (with "probability 1") the form (13) (or an equivalent form) for *any* value of x_c. Thus, the exact relation (6) is limited to a very small sub-class of KCM problems.

In the special case that $V(x,t) = V(x)$ (time independent), it follows from (7) and (8) that $W(-v) = W(v)$, an *even* function, and that $W = V_1 = V_2$. Thus, in this case, the KH model exactly related to the KCM problem must be symmetric and even. An example is the original KH model (1) with $L = K$. The existence of an exact relation between this model and the kicked harmonic oscillator (Hamiltonian (5) with $x_c = 0$) was mentioned in Ref. [10], but no detailed proof was given. The assumption of an even potential was made in Ref. [11] in order to derive several results concerning the quasienergy spectrum of generalized KH models. The case of an *odd* potential, $W(-v) = -W(v)$, was considered recently [12,16]. Both the classical and quantum dynamics in this case turn out to be significantly different than in the even case. We notice that, in the context of the full KCM problem, even and odd W's may often arise as a result of the variation of x_c. For example, if $V(x) = \cos(x)$, $W(v) = \cos(v)$ for $x_c = 0$, while $W(v) = \sin(v)$ for $x_c = \frac{1}{2}\pi$.

From the fact that the exact quantum relation (6) does not depend on $\hbar$, it follows from the correspondence principle that a similar relation must hold also classically [17]. As the minus sign in (6) is of pure quantum origin (the oscillator ground-state energy, see second equation in (10)), a natural guess for this classical relation is

$$M(K) = M_{KH}^{-2}(-\tfrac{1}{2}K), \tag{14}$$

where $M(K)$ and $M_{KH}(K)$ are the corresponding classical Poincaré maps from $t = -0$ to $t = T - 0$. In fact, a straightforward but tedious calculation, using the explicit expressions for $M(K)$ and $M_{KH}(K)$, shows that relation (14) indeed holds provided the following conditions are satisfied,

$$F(v,0) = -F(-v,\tfrac{1}{2}T) = F_1(v), \tag{15}$$

$$-F(-u,\tfrac{1}{4}T) = F(u,\tfrac{3}{4}T) = F_2(u), \tag{16}$$

where

$$F(v,t) = \partial W(v,t)/\partial v, \quad F_j(v) = \mathrm{d}V_j(v)/\mathrm{d}v,$$

$j = 1,2$, are the force functions. Clearly, conditions (15) and (16) are completely equivalent to (7) and (8). A result following immediately from relation (14) is that the hyperbolic-with-reflection periodic points of $M_{KH}(-\frac{1}{2}K)$ are ordinary hyperbolic periodic points of $M(K)$. It is also easy to show that, for $K \ll 1$ and to first order in K,

$$M(K) \approx M_{KH}(K), \tag{17}$$

provided again the conditions (15) and (16) are satisfied. relation (17) generalizes the approximation scheme given in Ref. [1] for the even potential $W(v) = -\cos(v)$. By combining relations (14) and (17), we get the interesting approximate relation

$$M_{KH}(K) \approx M_{KH}^{-2}(-\tfrac{1}{2}K). \tag{18}$$

Relation (18) may turn out to be useful in the investigation of the classical dynamics of generalized KH models in the important limit $K \to 0$.

We conclude with a discussion of the ordinary Harper model [14,18,19] in the light of our results. As it is well known, this model provides an approximate description of the problem of 2D Bloch electrons in a magnetic field within a tight-binding band $E(\boldsymbol{k}) = E_1(k_x) + E_2(k_y)$, where $\boldsymbol{k}$ is the quasimomentum. The Harper Hamiltonian [18] is obtained from $E(\boldsymbol{k})$ by the Peierls [20] substitution $\boldsymbol{k}\to\boldsymbol{\Pi}/\hbar$. Assuming, without loss of generality, that $\omega>0$ ($q>0$), this Hamiltonian may be expressed in terms of the u and v variables as $E(\boldsymbol{\Pi}/\hbar) = E_1(\omega u/\hbar) + E_2(\omega v/\hbar)$. As only the (u,v) degree of freedom appears in $E(\boldsymbol{\Pi}/\hbar)$, this Hamiltonian is obviously integrable. Thus, it can never give an exact description of the original problem of 2D Bloch electrons in a magnetic field, which involves the second degree of freedom of (x_c, y_c) and is generally nonintegrable. As pointed out in Refs. [14] and [19], the Harper model is only an approximate and consistent framework to account for the spectrum within a single tight-binding band or within a broadened Landau level. These spectra involve, effectively, only one degree of freedom. On the other hand, both the KH and KCM problems correspond to nonintegrable systems with 1.5 degrees of freedom, so that an exact relation between them cannot be excluded, in principle. In fact, it exists, as we have shown. In some special cases, also the Harper model can be exactly related to the KH and KCM problems, as shown in the Appendix.

The author is indebted to D. Shepelyansky for stimulating discussions and helpful comments. Useful discussions with M. Feingold and R.S. MacKay are gratefully acknowledged.

Appendix

We show here that the general Harper model (see definition above) can be exactly related to the KH and KCM problems when the magnetic flux Ba_xa_y through a unit cell (with lattice constants a_x and a_y) is an even integer multiple of the quantum of flux, hc/q. Consider the KH model (3) with $2KV_1(v)/T = E_2(\omega v/\hbar)$ and $2KV_2(u)/T = E_1(\omega u/\hbar)$, and let $\eta\equiv Ba_xa_y/(hc/q) = \omega a_xa_y/2\pi\hbar$. Since $E_1(k_x)$ ($E_2(k_y)$) is periodic with period $2\pi/a_x$ ($2\pi/a_y$), it follows that $V_1(v)$ ($V_2(u)$) is periodic with period a_x/η (a_y/η). By Fourier expanding the periodic functions $V_1(v)$ and $V_2(u)$ in powers of the phase-space translation operators $\exp(2i\pi\eta v/a_x)$ and $\exp(2i\pi\eta u/a_y)$, and using the well-known multiplication law [15] of these operators, it is easily verified that if η is an even integer one has the exact relation

$$U_H(K) = U_{KH}(K) = -U(K), \tag{19}$$

where

$$U_H(K) = \exp\{-2iK[V_1(v) + V_2(u)]/\hbar\}$$

is the Harper evolution operator in the time period T. In contrast with relation (6), however, relation (19) has no classical counterpart.

References

[1] G.M. Zaslavskii, M.Yu. Zakharov, R.Z. Sagdeev, D.A. Usikov and A.A. Chernikov, Sov. Phys. JETP 64 (1986) 294.
[2] G.M. Zaslavsky, R.Z. Sagdeev, D.A. Usikov and A.A. Chernikov, Weak chaos and quasi-regular patterns (Cambridge Univ. Press, Cambridge, 1991).
[3] A.J. Lichtenberg and B.P. Wood, Phys. Rev. A 39 (1989) 2153.
[4] P. Leboeuf, J. Kurchan, M. Feingold and D.P. Arovas, Phys. Rev. Lett. 65 (1990) 3076; Chaos 2 (1992) 125.
[5] R. Lima and D. Shepelyansky, Phys. Rev. Lett. 67 (1991) 1377.
[6] T. Geisel, R. Ketzmerick and G. Petschel, Phys. Rev. Lett. 67 (1991) 3635.
[7] R. Artuso, G. Casati and D. Shepelyansky, Phys. Rev. Lett. 68 (1992) 3826.
[8] R. Ketzmerick, G. Petschel and T. Geisel, Phys. Rev. Lett. 69 (1992) 695.
[9] R. Artuso, F. Borgonovi, I. Guarneri, L. Rebuzzini and G. Casati, Phys. Rev. Lett. 69 (1992) 3302.
[10] D. Shepelyansky and C. Sire, Europhys. Lett. 20 (1992) 95.
[11] I. Guarneri and F. Borgonovi, J. Phys. A 26 (1993) 119.
[12] I. Dana, Phys. Rev. Lett. 73 (1994) 1609.
[13] D.R. Grempel, R.E. Prange and S. Fishman, Phys. Rev. A 29 (1984) 1639.
[14] I. Dana and J. Zak, Phys. Rev. B 32 (1985) 3612, and references therein.
[15] A.M. Perelomov, Sov. Phys. Usp. 20 (1977) 703, and references therein.
[16] I. Dana and M. Amit, to be published.
[17] D. Shepelyansky, private communication.
[18] P.G. Harper, Proc. R. Soc. A 68 (1955) 874.
[19] G. Petschel and T. Geisel, Phys. Rev. Lett. 71 (1993) 239.
[20] R.E. Peierls, Z. Phys. 80 (1933) 763.