Strong Variation of Global-Transport Properties in Chaotic Ensembles

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Abstract. Chaotic transport is studied for Hamiltonians H in which one coordinate, say q, is *cyclic* (i.e., it does not appear in H), leading to the conservation of the conjugate coordinate ("momentum" p). It is assumed that the dynamics depends ***nontrivially*** on the "parameter" p in H. As a consequence, one expects to observe a variation of the global-transport properties, both normal and anomalous, in a generic chaotic ensemble that exhibits all values of p. By considering the realistic model system of charged particles interacting with an electrostatic wave-packet in a uniform magnetic field, it is shown that this variation can be actually quite strong. This finding may have applications to "filtering" sub-ensembles with well-defined values of p.

Hamiltonian chaos (see, e.g., MacKay and Meiss 1987 and references therein) is a unique phenomenon in that it generically appears interleaved with ordered/stable motions on all scales of phase space (Meiss 1986; Umberger and Farmer 1985), leading to long-time correlations (Karney 1983; Meiss and Ott 1986) and quasiregularity (Dana 1993) in the chaotic motion. A fundamental question is then to what extent the transport due to the deterministic chaos resembles that associated with a truly probabilistic random process, such as Brownian motion (Chirikov 1979). This question has been investigated extensively during the last two decades, mainly for systems which can be described by area-preserving maps. A globally diffusive transport, $\langle R^2 \rangle = 2Dt$ ($\langle\ \rangle$ denotes initial-ensemble average, $\mathbf{R}$ is some radius vector in the phase space, and D is the diffusion coefficient), is often observed numerically (see, e.g., Chirikov 1979; Dana and Fishman 1985) but occurs rigorously only in very special cases (Cary and Meiss 1981). The self-similar islands-around-islands hierarchy in phase space [Meiss 1986; Zaslavsky et al. (1997)] should be responsible to the anomalous global diffusion, $\langle R^2 \rangle \propto t^\mu$ $(0 < \mu < 2)$ [Shlesinger et al. 1993; Zumofen and Klafter 1994; Zaslavsky et al. (1997); Afraimovich and Zaslavsky (1997)], which may be described by Lévy random-walk processes (Shlesinger et al. 1993; Zumofen and Klafter 1994).

Because of the complex phase-space structure of a generic Hamiltonian system, chaotic transport is usually quite inhomogeneous *locally* (Karney 1983; MacKay et al. 1984; Dana et al. 1989; Afanasiev et al. 1991). In this paper, we show that one can also observe a high inhomogeneity in the *global*-transport properties due to the following simple scenario. Consider a Hamiltonian H in which one coordinate, say q, is *cyclic*, i.e., it does not appear in H. The conjugate coordinate ("momentum"), p, is then a constant of the motion and appears in H as a "parameter", $H = H(\mathbf{R},\ t;\ p)$. Here $\mathbf{R}$ denotes all the other phase-space coordinates and, for the sake of generality, a dependence on time t is included. Our crucial assumption is that the dynamics in the $\mathbf{R}$ phase space depends *nontrivially* on the "parameter" p. Now, since p is actually a coordinate, a generic, realistic ensemble of particles will exhibit all values of p. Such an ensemble can be divided into sub-ensembles characterized by well-defined values of p. The assumption above then implies that different sub-ensembles will be characterized by different global-transport properties, e.g., a normal-diffusion coefficient $D(p)$ or an anomalous-diffusion exponent $\mu(p)$. As a result, a variation of these properties throughout the entire ensemble will be observed.

We show here that this variation can be actually quite strong by considering the realistic model system of charged particles interacting with an electrostatic wave-packet in a uniform magnetic field. This system is described by the Hamiltonian

$$H = \Pi^2/(2M) + KV(kx,\ t)\ , \tag{1}$$

where $\Pi = \mathbf{p} - e\mathbf{A}/c$ is the kinetic momentum of a particle with charge e and mass M in a uniform magnetic field $\mathbf{B}$ (along the z-axis), K is a parameter, $\mathbf{k}$ is the wave-vector (in the x-direction), and V is a general function describing the electrostatic wave-packet. This function is periodic in both kx (with period 2π) and time t (with period T). Without loss of generality, the values of M and k will be both set to 1 from now on.

To see that (1) is a Hamiltonian of the kind described above, let us express it using the natural degrees of freedom in a magnetic field. These are given by the conjugate pairs $(x_c,\ y_c)$ (coordinates of the center of a cyclotron orbit) and $(\Pi_x,\ \Pi_y)$, see Johnson and Lippmann (1949). Defining $u = \Pi_x/|\omega|$, $v = \Pi_y/\omega$, where $\omega = eB/c$ is the cyclotron frequency, and using the relation $x_c = x + \Pi_y/\omega = x + v$ (easily derivable from simple geometry), (1) can be rewritten as follows

$$H = \omega^2(u^2 + v^2)/2 + KV(x_c - v,\ t)\ . \tag{2}$$

It is now clear that y_c is cyclic in H, so that it corresponds to the coordinate q above. The conserved "momentum" p is then x_c.

In what follows, we shall assume the simple wave-packet

$$V(x,\ t) = -\cos x \sum_{s=-\infty}^{\infty} \delta(t - sT)\ ,$$

reducing (2) to the Hamiltonian of a kicked harmonic oscillator. The latter system has been investigated extensively by Zaslavsky et al. (1986) (see the review article by Zaslavsky 1991) who assumed, however, the very specific value $x_c = 0$ in (2). These investigations have led to the discovery of the well-known properties of this system. Since the harmonic oscillator is degenerate (linear in the action), the nonlinear perturbation in (2) is strong (in the sense of KAM theory) for all values of K, especially under resonance conditions, $\omega T = 2\pi m/n$ (m and n are coprime integers). One then expects, on the basis of general arguments, that unbounded chaotic motion of $(u,\ v)$ should exist for arbitrarily small values of K in the resonance case. This motion is observed to take place diffusively on a "stochastic web" [see Fig. 1(a)], analogous in some aspects to the Arnol'd web. For $n = 3,\ 4,\ 6$, the web has crystalline symmetry (triangular, square, hexagonal), while for all other values of $n > 4$ it has quasicrystalline symmetry.

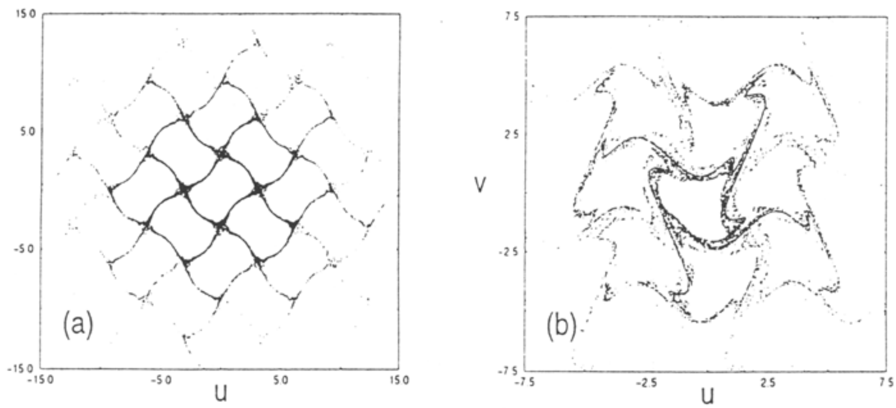

Fig. 1. Portions of the stochastic webs for $m/n = 1/4$, $K = 1.4$, and (a) $x_c = 0$, (b) $x_c = \pi/2$. Each plot contains 40 000 points of chaotic orbits, generated by iterating 100 times an ensemble of 20 × 20 initial conditions near the origin with the map corresponding to (2). Notice that the diffusion rate in case (b) is slower than in case (a). Without loss of generality, the value of ω in (2) is set to 1 in this paper.

The need to consider general values of x_c has been pointed out only recently by Dana and Amit (1995), who developed a general formalism for calculating the normal-diffusion coefficient $D(x_c)$ for Hamiltonian (2) as a function of x_c. Here $D(x_c)$ is defined, under resonance conditions $\omega T = 2\pi m/n$, by

$$D(x_c) = \lim_{s\to\infty} \frac{1}{2sn} \langle R_{sn}^2 \rangle_{E(x_c)} \ , \qquad R_s^2 \equiv (u_s - u_0)^2 + (v_s - v_0)^2 \tag{3}$$

(assuming the limit exists), where $(u_s,\ v_s)$, s integer, are the values of $(u,\ v)$ at times $sT - 0$ and the average $\langle\ \rangle$ is taken over a sufficiently large sub-ensemble $E(x_c)$ of initial conditions $(u_0,\ v_0)$ at fixed x_c. As already emphasized by Dana and Amit (1995), the average D_{av} of $D(x_c)$ over x_c is of practical importance, since in an experiment one usually measures the average diffusion rate of a generic ensemble, exhibiting all the values of x_c. Here we shall focus on the dependence of the global-transport properties on x_c. An impressive example showing this dependence was given, apparently for the first time, by Dana (1994) in a quantum-chaos context: for the $n = 4$ web (square crystalline symmetry), and for small K, the diffusion rate for $x_c = \pi/2$ is much slower than that for $x_c = 0$. Traces of this phenomenon can be observed already for K not very small, as shown in Fig. 1. Later, Pekarsky and Rom-Kedar (1997) have shown that for small K the $n = 4$ web undergoes a dramatic structural change, mediated by a sequence of bifurcations, as x_c is varied from $x_c = 0$ to $x_c = \pi/2$ (this can also be seen in Fig. 1). They showed that the width of the stochastic layer of the web is proportional to $\exp(-\pi^2/K^\epsilon)$, where $\epsilon = 1$ for $x_c = 0$ and $\epsilon = 2$ for $x_c = \pi/2$. This explains the strong difference in the diffusion rate in the two cases for small K, observed by Dana (1994).

Analytical expressions approximating $D(x_c)$ to high accuracy for K sufficiently large can be obtained using the formalism of Dana and Amit (1995). For example, for the $n = 4$ web we find

$$D(x_c) \approx K^2 \left\{ \frac{1}{4} + \frac{J_0(K)}{2}\cos(2x_c) + \frac{1}{2}\sum_{r=-\infty}^{\infty} \exp(-2irx_c) \times \right.$$
$$\left. \left[(-1)^r J_0(rK) J_r^2(K) - J_2(rK) J_r^2(K) \cos(4x_c) \right] \right\} , \qquad (4)$$

where $J_r(K)$ is a Bessel function. For K sufficiently large, the expression in (4) can be simplified by identifying the dominant terms in the sum over r.

A variation of the global-transport properties, which is much stronger than that in the normal-diffusion case [e.g., $D(x_c)$ in (4)], can be observed when anomalous diffusion is present,

$$\left\langle R_{sn}^2 \right\rangle_{E(x_c)} \propto s^{\mu(x_c)} , \qquad (5)$$

where R_s^2 is defined by (3) and $\mu(x_c)$ is the anomalous-diffusion exponent, $\mu(x_c) \neq 1$. "Superdiffusion", with $1 < \mu(x_c) < 2$, can be observed for sufficiently large values of K in the case of the crystalline webs ($n = 3,\ 4,\ 6$). In this case, the translational symmetry allows for the existence of generalized periodic orbits, the "accelerator modes". Their defining equations are

$$u_{sn+ln} = u_{sn} + 2\pi j_1 , \qquad v_{sn+ln} = v_{sn} + 2\pi j_2 , \qquad (6)$$

for all integers s, where l is the minimal period and $2\pi(j_1,\ j_2)$ is a lattice vector characterizing the accelerator mode. If sn is replaced by $sn + r$, $r = 1, ..., ln - 1$ (corresponding to the "other" points of the periodic orbit), Eqs. (6) will be satisfied with $(j_1,\ j_2)$ replaced by $O^r(j_1,\ j_2)$, where O is a rotation by an angle $\alpha = 2\pi m/n$. If the accelerator mode is linearly stable, each point of it is usually surrounded by a stability island. All the points within an island move essentially (i.e., on a sufficiently large scale) according to Eqs. (6), leading to "acceleration",

$R^2_{sln} \propto s^2$ (i.e., $\mu = 2$). On the other hand, points in the chaotic region (stochastic layer of the web) will "stick" near the boundaries of the islands, following their accelerating motion for a long time interval, and are ejected afterwards back inside the chaotic region. After some time, they will eventually stick again near the boundaries of the islands. This process explains figuratively the origin of the global superdiffusion with an exponent μ taking values between $\mu = 1$ (corresponding to the normal diffusion expected in a strongly-chaotic regime or in the absence of accelerator islands) and $\mu = 2$ (corresponding to acceleration within the islands). A quantitative explanation of superdiffusion and a general relation between μ and the self-similarity properties of accelerator islands have been given recently by Afraimovich and Zaslavsky (1997) [see also the recent review article by Zaslavsky et al. (1997)].

In the case of our system, the crucial observation is that, for a given value of K, accelerator islands may exist only in some intervals of x_c. In these intervals, the characteristics of the islands usually vary strongly with x_c. This is shown in Fig. 2 for the $n = 4$ web at $K = 3.25$. For this value of K, we were able to find only accelerator modes of minimal period $l = 1$ with ($j_1 = 1$, $j_2 = 0$) and ($j_1 = 1$, $j_2 = 1$) [recall the definition (6)]. These modes exist only in the x_c-intervals covered by the several curves in Fig. 2. The modes are linearly stable and give rise, usually, to accelerator islands only if the trace of their linearity-stability matrix is between -2 and 2 (the two horizontal dashed lines in Fig. 2). Fig. 3 shows an enlargement of Fig. 2 in the main interval of x_c where the

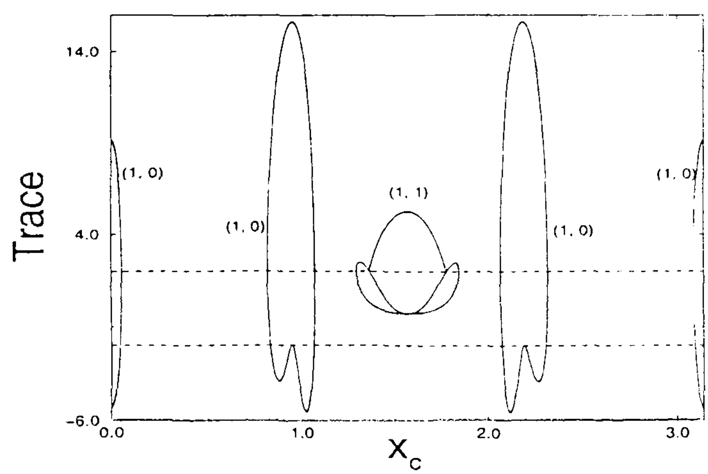

Fig. 2. Trace of the linear-stability matrix as a function of x_c for the accelerator modes with minimal period $l = 1$ in the case of $n = 4$ and $K = 3.25$. These modes exist only in the intervals of x_c covered by the several curves, and are linearly stable only if the trace is between -2 and 2 (the two horizontal dashed lines). The label (1, 0) or (1, 1) near each curve is the type (j_1, j_2) of the corresponding mode.

(1, 1) accelerator mode exists. We also plot here a properly normalized area $S(x_c)$ of the corresponding accelerator island as a function of x_c. Obviously, $S(x_c)$ vanishes for x_c outside the interval.

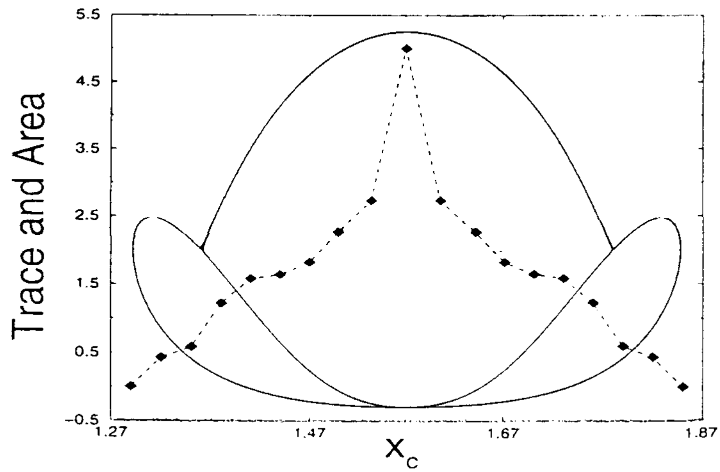

Fig. 3. An enlargement of Fig. 2 in the main x_c-interval of existence of the (1, 1) accelerator mode. The dashed line with diamond symbols gives the area of the corresponding accelerator island, in units such that the maximum value of the area, at $x_c = \pi/2$, is 5. The area was actually computed only for $x_c \leq \pi/2$, and the reflection symmetry around $x_c = \pi/2$ was used to complete the plot for $x_c > \pi/2$.

We have performed an accurate calculation of the anomalous-diffusion exponent $\mu(x_c)$ ($n = 4$, $K = 3.25$) for the same values of x_c used to plot the curve of $S(x_c)$ in Fig. 3. This calculation was made as follows. For a given value of x_c, a large ensemble of 400×400 initial conditions, uniformly distributed in the $2\pi \times 2\pi$ unit cell of the web, was iterated 1219680 times with the map corresponding to (2). Initial conditions inside accelerator islands were easily identified by their accelerating motion, and were removed from the ensemble. The remaining ensemble, $E(x_c)$, should consist then entirely of initial conditions inside the chaotic region. Indeed, we have found that for times $t = sn < 1219680$ the ensemble $E(x_c)$ evolves reasonably well according to the anomalous-diffusion law (5). The anomalous-diffusion exponent $\mu(x_c)$ was determined from the best fit of the function $f(s) = Bs^{\mu}$ to $\langle R_{sn}^2 \rangle_{E(x_c)}$. The results are shown in Fig. 4. The strong oscillatory variation of μ with x_c, from $\mu \approx 1$ (i.e., nearly normal diffusion) to $\mu \approx 1.5$, is quite remarkable! Notice that the oscillatory behavior of $\mu(x_c)$ is quite different from the monotonous one of $S(x_c)$ (the area of the accelerator island in Fig. 3). In particular, the maximal value of $\mu(x_c)$ is not attained at $x_c = \pi/2$, as in the case of $S(x_c)$. In fact, $\mu(x_c)$ is really determined not by $S(x_c)$

but by the self-similarity properties of the accelerator islands [Afraimovich and Zaslavsky (1997); Zaslavsky et al. (1997)].

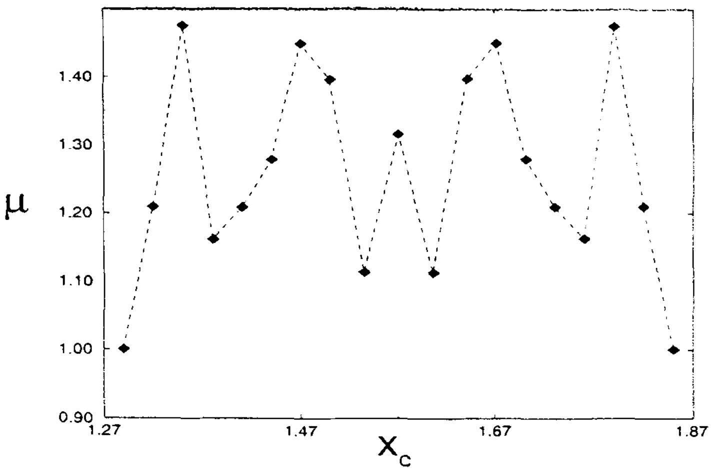

Fig. 4. Anomalous-diffusion exponent $\mu(x_c)$ for $n = 4$ and $K = 3.25$, calculated as explained in the text. As for the area curve in Fig. 3, $\mu(x_c)$ was actually computed only for $x_c \leq \pi/2$, and the reflection symmetry around $x_c = \pi/2$ was used to complete the plot for $x_c > \pi/2$.

In conclusion, we have shown that global-transport properties, such as the normal-diffusion coefficient D and the anomalous-diffusion exponent μ, can vary throughout a chaotic ensemble due to a general and simple scenario. This has been illustrated by a realistic model system of charged particles interacting with an electrostatic wave-packet in a uniform magnetic field. For this system, we have found that the variation of the global-transport properties, mainly the anomalous ones, can be remarkably strong. We expect that this finding should have experimental applications to "filtering" or preparing sub-ensembles characterized by well-defined values of the conserved momentum, e.g., x_c. This can be easily accomplished, for example, by considering electrostatic wave-packets depending on a "phase" ϕ, i.e., $V = V(x - \phi,\ t)$, and by adjusting ϕ so the maximal transport rate is attained at the desired value of x_c. Other theoretical aspects of the problem considered in this paper will be studied in future publications.

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References

Afanasiev, V. V., Sagdeev, R. Z., Zaslavsky, G. M. (1991): CHAOS **1**, 143.

Afraimovich, V., Zaslavsky, G. M. (1997): Phys. Rev. E **55**, 5418.

Cary, J. R., Meiss, J. D. (1981): Phys. Rev. A **24**, 2664.

Chirikov, B. V. (1979): Phys. Rep. **52**, 263.

Dana, I. (1993): Phys. Rev. Lett. **70**, 2387.

Dana, I. (1994): Phys. Rev. Lett. **73**, 1609.

Dana, I., Amit, M. (1995): Phys. Rev. E **51**, R2731.

Dana, I., Fishman, S. (1985): Physica (Amsterdam) **17D**, 63.

Dana, I., Murray, N. W., Percival, I. C. (1989): Phys. Rev. Lett. **62**, 233.

Johnson, M. H., Lippmann, B. A. (1949): Phys. Rev. **76**, 828.

Karney, C. F. F. (1983): Physica (Amsterdam) **8D**, 360.

MacKay, R. S., Meiss, J. D. (1987): *Hamiltonian Dynamical Systems* (Adam Hilger, Bristol).

MacKay, R. S., Meiss, J. D., Percival, I. C. (1984): Physica (Amsterdam) **13D**, 55.

Meiss, J. D. (1986): Phys. Rev. A **34**, 2375.

Meiss, J. D., Ott, E. (1986): Physica (Amsterdam) **20D**, 387.

Pekarsky, S., Rom-Kedar, V. (1997): Phys. Lett. A **225**, 274.

Umberger, D. K., Farmer, J. D. (1985): Phys. Rev. Lett. **55**, 661.

Shlesinger, M. F., Zaslavsky, G. M., Klafter J. (1993): Nature **363**, 31.

Zaslavsky, G. M. (1991): CHAOS **1**, 1.

Zaslavsky, G. M., Edelman, M., Niyazov, B. A. (1997): CHAOS **7**, 159.

Zaslavsky, G. M., Zakharov, M. Yu., Sagdeev, R. Z., Usikov, D. A., Chernikov, A. A. (1986): Sov. Phys. JETP **64**, 294.

Zumofen, G., Klafter, J. (1994): Europhys. Lett. **25**, 565.