

Statistical approach to quantum chaotic ratchets: First results and open problems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2011 J. Phys.: Conf. Ser. 285 012048

(<http://iopscience.iop.org/1742-6596/285/1/012048>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 77.126.0.1

The article was downloaded on 12/04/2011 at 12:37

Please note that [terms and conditions apply](#).

Statistical approach to quantum chaotic ratchets: First results and open problems

Itzhack Dana

Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

E-mail: dana@mail.biu.ac.il

Abstract. This paper is a brief review of a new approach to the quantum-chaotic ratchet effect, introduced recently to address for the first time the *sensitivity* of the effect to the *initial state* in a *global* fashion. This is done by studying *statistical* properties of the ratchet current over well-defined sets of initial states. First results concern the semiclassical full-chaos regime, where the current is strongly sensitive to the initial state. Natural initial states in this regime are those that are *phase-space uniform* with the *maximal possible* resolution of one Planck cell. General arguments, for a class of paradigmatic model systems and for special quantum-resonance values of a scaled Planck constant \hbar , predict that the distribution of the momentum current over all such states is a zero-mean Gaussian with variance $\sim D\hbar^2/(2\pi^2)$, where D is the chaotic-diffusion coefficient. This prediction is well supported by extensive numerical evidence. The average strength of the effect, measured by the variance above, is *significantly larger* than that for the usual momentum states and other states. Open problems, concerning extensions of these first results in different directions, are discussed.

1. Introduction

Classical [1–6] and quantum [3, 6–14] Hamiltonian ratchets have attracted a considerable theoretical interest during the last decade. Also, several kinds of quantum ratchets have been experimentally realized using atom-optics methods with cold atoms or Bose-Einstein condensates [15–18]. The classical Hamiltonian ratchet effect is a directed current in the chaotic region generated by an unbiased force (having zero mean in space and/or time) and due to some spatial and/or temporal asymmetry [1–6]. This is analogous to the ordinary ratchet effect [19], but with deterministic chaos replacing the usual noisy environment. Also, the important ingredient of dissipation in ordinary ratchets is absent in Hamiltonian ratchets.

A basic general result for Hamiltonian dynamics under an unbiased force is that the average current of an initially *uniform* ensemble of particles in phase space is zero [2,3]. As a consequence, a completely chaotic system carries essentially no ratchet current. On the other hand, the corresponding quantized system can feature significant ratchet effects [7–17]. An important problem is to understand the nature of these full-chaos quantum effects in a semiclassical regime, in particular how precisely they vanish, as expected, in the classical limit. Standard studies of quantum chaotic ratchets have mainly focused on the impact of several kinds of asymmetries on the quantum directed current from a *fixed* initial state. It is, however, well established that the current is *sensitive* to the initial state [3, 9, 11–14, 17] and this sensitivity is expected to be especially high in a semiclassical full-chaos regime, reflecting the exponential sensitivity of chaotic motion to initial conditions. Thus, to get a comprehensive understanding of the quantum

ratchet effect, it is necessary to adopt a more *global* approach, *not* limited to a single initial state.

Such approach was introduced in recent works [13,14]. It considers for the first time *statistical* properties of the ratchet current over whole sets of initial states with well-defined characteristics. The first results of this new approach concern the semiclassical full-chaos regime of simple but paradigmatic model systems, the generalized kicked Harper models (KHMs) [20–27] with Hamiltonian

$$\hat{H} = L \cos(\hat{p}) + KV(\hat{x}) \sum_{t=-\infty}^{\infty} \delta(t' - t), \quad (1)$$

where L and K are parameters, \hat{x} and \hat{p} are scaled position and momentum operators, $V(\hat{x})$ is a general 2π -periodic potential, t' is time, and t is the integer time labeling the kicks. Generalized KHMs such as (1) describe several realistic systems [12,24,25,27], in particular they are exactly related [24,25] to kicked harmonic oscillators, which are experimentally realizable by atom-optics methods [28], and to kicked charges in a magnetic field [25]. Recently [10], the systems (1) were shown to exhibit generically a robust quantum *momentum* current (ratchet acceleration) under full-chaos conditions; this quantum ratchet effect is, of course, much more significant than the usual one, i.e., a quantum position current (velocity). The fixed initial state was chosen, as in other works, as a zero-momentum state.

This paper provides, in section 2, a detailed summary of the first results of the new statistical approach [13,14] and, in section 3, a discussion of open problems concerning extensions of these results in several different directions.

2. Detailed summary of first results

2.1. Maximally uniform states in phase space and approximations of them

In the statistical approach to the quantum-chaotic ratchet effect [13,14], natural initial states for the semiclassical regime are identified as those that are analogous as much as possible to a phase-space uniform ensemble, for which classical ratchet effects are totally absent. These are states which are *uniform* in *phase space* with the *maximal possible* resolution of one Planck cell. Such uniformity is not featured by, e.g., the usual momentum state which is uniform in position but is infinitely localized in momentum.

The maximally uniform states (MUSs) in phase space are simultaneous eigenstates of the translation operators $\hat{T}_x(a) = \exp(i\hat{p}a/\hbar)$ and $\hat{T}_p(b) = \exp(-i\hat{x}b/\hbar)$ shifting \hat{x} and \hat{p} ($[\hat{x}, \hat{p}] = i\hbar$) by a and b , respectively, where a and b define a Planck cell, $ab = h = 2\pi\hbar$. This is the minimal value of the area ab for which $\hat{T}_x(a)$ and $\hat{T}_p(b)$ commute. Then, the simultaneous eigenstates (the MUSs) $|\psi\rangle$ are the well known “ kq ” states [29], given in both the position and momentum representation by “delta combs”:

$$\langle x | \psi_{\mathbf{w}} \rangle = \sum_{n=-\infty}^{\infty} e^{2\pi i n w_2 / b} \delta(x - w_1 - na), \quad (2)$$

$$\langle p | \psi_{\mathbf{w}} \rangle = \sum_{n=-\infty}^{\infty} e^{-2\pi i n w_1 / a} \delta(p - w_2 - nb), \quad (3)$$

where $\mathbf{w} = (w_1, w_2)$ range in the Planck cell, $0 \leq w_1 < a$, $0 \leq w_2 < b$. The states (2) or (3) for all \mathbf{w} form a complete and orthogonal set [29].

Natural approximations of the MUSs $|\psi_{\mathbf{w}}\rangle$, denoted in what follows by $|\psi_{\mathbf{w}}^{(B)}\rangle$ for integer B , correspond to truncations of the infinite sum in (3):

$$\langle p | \psi_{\mathbf{w}}^{(B)} \rangle = \sum_{n=-B}^B e^{-2\pi i n w_1 / a} \delta(p - w_2 - nb). \quad (4)$$

The states (4) are superpositions of the $2B + 1$ momentum states $|p = w_2 + nb\rangle$, $|n| \leq B$. In particular, $|\psi_{\mathbf{w}}^{(0)}\rangle$ are just the usual momentum states with $p = w_2$. Superpositions of two momentum states were recently used in experimental realizations of quantum-resonance ratchets [16,17]. The states (4) can be experimentally prepared for at least $B \approx 10$ [30].

2.2. Exact formulas for the quantum momentum current (acceleration)

For the systems (1), the momentum-current operator \hat{I} can be formally defined in the Heisenberg picture as $\hat{I} = \lim_{t \rightarrow \infty} \hat{U}^{-t} \hat{p} \hat{U}^t / t$ (for integer time t), where $\hat{U} = \exp[-L \cos(\hat{p})/\hbar] \exp[-KV(\hat{x})/\hbar]$ is the KHM one-period evolution operator. Then, the momentum current $I(\mathbf{w})$ for initial state (2) is the expectation value of \hat{I} in (2). An exact formula for $I(\mathbf{w})$ can be derived for rational values q/N (q and N are coprime integers) of $\hbar/(2\pi)$, corresponding to quantum resonances. To present this formula, we first recall that the eigenstates of the operator \hat{U} , i.e., the quasienergy (QE) or Floquet states, can be written as [26]

$$|\Psi_{j,\mathbf{w}}\rangle = \sum_{m=0}^{N-1} \phi_j(m; \mathbf{w}) |\psi_{\mathbf{w},m}\rangle, \quad (5)$$

$j = 1, \dots, N$, where, for $m = 0, \dots, N-1$, $\phi_j(m; \mathbf{w})$ are N coefficients and $|\psi_{\mathbf{w},m}\rangle$ are N states (2) on a lattice of N points (\mathbf{w}, m) in the Planck cell. The corresponding QE spectrum consists, at fixed \mathbf{w} , of N levels $E_j(\mathbf{w})$ and, as \mathbf{w} varies continuously in the Planck cell, these levels broaden into N QE bands. Then

$$I(\mathbf{w}) = -\hbar \sum_{j=1}^N |\phi_j(0; \mathbf{w})|^2 \frac{\partial E_j(\mathbf{w})}{\partial w_1}. \quad (6)$$

The momentum current $I_B(\mathbf{w})$ for initial state (4) can be exactly related to the current (6):

$$I_B(\mathbf{w}) = \int_0^a dw'_1 g_B^2(w'_1 - w_1) I(w'_1, w_2), \quad (7)$$

where

$$g_B(w_1) = \frac{1}{\sqrt{(2B+1)a}} \sum_{n=-B}^B \exp(2\pi i n w_1 / a). \quad (8)$$

In particular, the momentum current for initial momentum states $|p\rangle$ ($B = 0$, $g_0(w_1) = 1/\sqrt{a}$) is just $I_p = \int_0^a dw_1 I(w_1, w_2 = p)/a$.

2.3. Mean momentum current is exactly zero

We now consider a first statistical property of the currents (6) or (7): The mean value of such a current over the corresponding states, i.e., over \mathbf{w} , is *exactly zero*. To show this in the case of the current (6), one assumes, for definiteness, that the N eigenvalues $\exp[-iE_j(\mathbf{w})]$ of \hat{U} for $\hbar = 2\pi q/N$ are nondegenerate at fixed \mathbf{w} . This implies, using (5) and considerations similar to those in work [26], that: (a) Each QE band $E_j(\mathbf{w})$ is periodic in both w_1 and w_2 with period $2\pi/N$. (b) $|\phi_j(0; \mathbf{w})|$ is periodic in w_1 and w_2 with periods defining a unit cell \mathcal{C} q times smaller than the Planck cell. (c) $|\phi_j(m; \mathbf{w})| = |\phi_j(0; (\mathbf{w}, m))|$. It follows from (a), (b), and (6) that $I(\mathbf{w})$ is periodic in \mathbf{w} with unit cell \mathcal{C} . Using (a), (c), (6), and the normalization $\sum_{m=0}^{N-1} |\phi_j(m; \mathbf{w})|^2 = 1$, we get that $\int_0^{2\pi/N} dw_1 \sum_{m=0}^{N-1} I(w_1, w_2, m) = 0$ at any fixed w_2 . This result specifies general mixtures of the pure states (2) carrying a zero mean current. In particular, it implies that $I(\mathbf{w})$ has zero mean, $\langle I \rangle = \int_{\mathcal{C}} d\mathbf{w} I(\mathbf{w}) = 0$.

Using this fact and formula (7) with (8), it follows that the mean value of the current $I_B(\mathbf{w})$ is also exactly zero.

2.4. Semiclassical estimate of the quantum momentum-current distribution in simple cases

For the time being, the main result of the statistical approach is an estimate of the distribution $\Gamma(I)$ of the quantum momentum current $I(\mathbf{w})$ for the systems (1) in a semiclassical regime ($\hbar \ll 1$) of strong chaos (large K and L). According to this estimate, in the simple cases of $\hbar = 2\pi/N$ (i.e., $q = 1$, with $\hbar \ll 1$ corresponding to $N \gg 1$), $\Gamma(I)$ is approximately a zero-mean Gaussian with a well-defined simple variance:

$$\Gamma(I) \sim \frac{1}{\sqrt{2\pi}\Delta I} \exp\left[-\frac{I^2}{2(\Delta I)^2}\right], \tag{9}$$

$$(\Delta I)^2 = \langle I^2 \rangle \sim \frac{D\hbar^2}{2\pi^2} = \frac{2D}{N^2}, \tag{10}$$

where D is the classical chaotic-diffusion coefficient. Using relation (7), one can also derive an exact inequality concerning the variances of $I_B(\mathbf{w})$ and $I(\mathbf{w})$:

$$(\Delta I_B)^2 = \langle I_B^2 \rangle \leq (\Delta I)^2 = \langle I^2 \rangle. \tag{11}$$

The requirement $q = 1$ for the validity of the estimate (9) with (10) may be understood as follows. The classical KHM map corresponding to (1) is strictly 2π -periodic in both x and p . Thus, the torus $T^2: 0 \leq x, p < 2\pi$ is a reduced phase space for the classical system. On the other hand, the quantum KHM map, i.e., the one-period evolution operator \hat{U} above, is invariant under phase-space translations which commute only if they define a “quantum” torus T_Q^2 not smaller than q times the classical one T^2 . Thus, a classical-quantum correspondence for $\hbar \ll 1$ can be most easily established in the case of $q = 1$ with $T_Q^2 = T^2$ [20, 26]. Only in this case, as in the simple derivation of the estimate (9) with (10) in work [14], one does not have to consider tunneling effects between the q tori T^2 comprising T_Q^2 for $q > 1$.

For $q = 1$, one can choose in (2) $a = 2\pi/N_1$ and $b = 2\pi/N_2$, where N_1 and N_2 are integers satisfying $N_1N_2 = N$. Then, T^2 is precisely tiled by N Planck cells, which is very convenient. The results presented from now on are for $q = 1$ and the simple factorization $N_1 = N, N_2 = 1$.

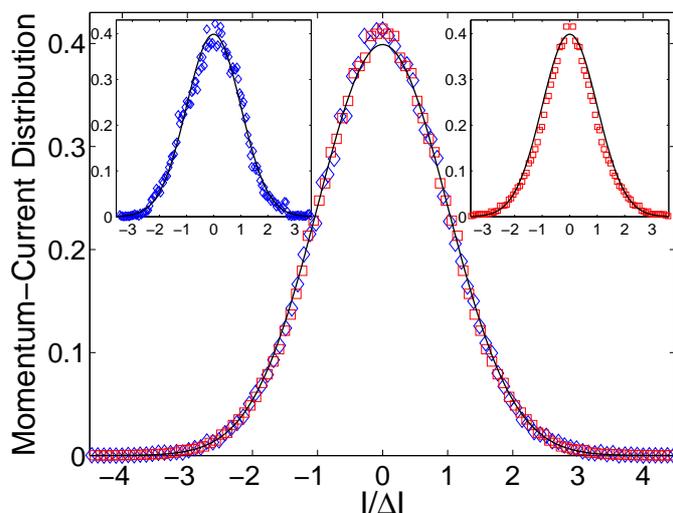


Figure 1. Distributions of the normalized quantum momentum current $I/\Delta I$ in different cases. See text for details.

Figure 1 show distributions of $I(\mathbf{w})/\Delta I$ (main plots), calculated using the exact formula (6) for $N = 121$ in two extreme cases of fully chaotic systems (1) with $K = 15$: The symmetric case

“*S*” with $V(x) = \cos(x)$ and $L = K$ (red squares, $\Delta I = 0.086$) and the strongly asymmetric case “*A*” with $V(x) = \cos(x) + \sin(2x)$ and $L = K/2$ (blue diamonds, $\Delta I = 0.214$). The latter case was studied in work [10] for a zero-momentum initial state. The insets show the distribution of $I/\Delta I$ over momentum states in case *A* with $K = 20$ (left inset, $\Delta I = 0.043$) and over low-order ($B = 2$) approximating states (4) in case *S* with $K = 15$ (right inset, $\Delta I = 0.026$). The solid line in all plots is a zero-mean Gaussian with variance 1. We see that this line provides a good fit to all the distributions, including those in the insets, which are over states $|\psi_{\mathbf{w}}^{(B)}\rangle$ significantly different from the MUSs $|\psi_{\mathbf{w}}\rangle$. This may be due to the relatively simple relation (7) between the currents associated with $|\psi_{\mathbf{w}}^{(B)}\rangle$ and $|\psi_{\mathbf{w}}\rangle$.

The origin of quantum ratchet effects in the symmetric case *S* in Figure 1 is the same as that already established in recent theoretical [11] and experimental [17] works on quantum-resonance ratchets: This is a relative asymmetry caused by the non-coincidence of the symmetry centers of a symmetric potential with those of a symmetric initial state for the assumed quantum-resonance values $2\pi/N$ of $\hbar = a$. The potential $V(x) = \cos(x)$ in case *S* has symmetry centers at $x = 0, \pi$ while the state (2) has them at $x = w_1, w_1 + a/2$. Thus, for generic values of w_1 , $I(\mathbf{w}) \neq 0$.

2.5. Dependence of variances of quantum momentum currents on parameters

We now compare the results (10) and (11), concerning the dependence of variances on several parameters, with numerical data obtained using the exact formulas (6) and (7). Consider first the naturally normalized variance $R \equiv N^2(\Delta I)^2/(2D_{\text{ql}})$, where D_{ql} is the “quasilinear” value of the diffusion coefficient D , corresponding to the lowest-order term in an expansion of D in force-force correlations; for sufficiently strong chaos, D is very close to D_{ql} . The semiclassical estimate for the variance in Eq. (10) would imply that $R \approx D/D_{\text{ql}}$. Indeed, Figure 2 shows a reasonably good agreement between R and D/D_{ql} versus K in both cases *S* and *A*. Discrepancies arise mainly around peaks of D/D_{ql} , especially the peak near $K \approx 6.5$ in case *S*, due to a small accelerator-mode island. Thus, for general large K with $R \approx 1$, ΔI increases almost like $\sqrt{D_{\text{ql}}}$ (i.e., linearly in K).

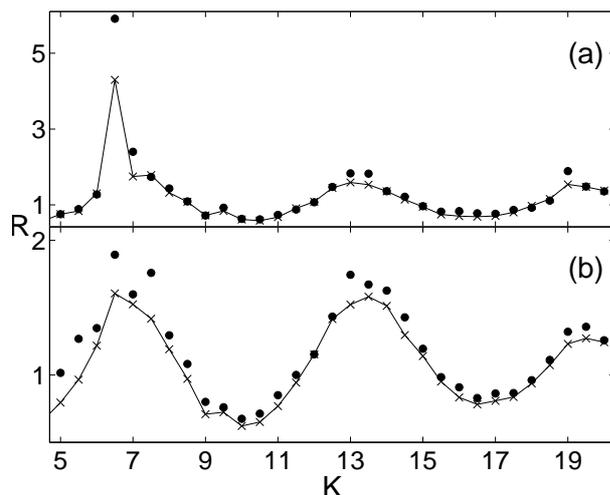


Figure 2. Filled circles: The quantity $R = N^2(\Delta I)^2/(2D_{\text{ql}})$ versus K for $N = 2\pi/\hbar = 121$ in cases *S* [(a), $D_{\text{ql}} = K^2/4$] and *A* [(b), $D_{\text{ql}} = 5K^2/4$] defined in the text. Crosses joined by a line: D/D_{ql} versus K .

Figure 3 shows loglog plots of ΔI versus N in cases *S* and *A*. The results agree very well with the N^{-1} behavior predicted by (10). The inset shows plots of $\Delta I_B/\Delta I$ versus B in the two cases. We see that ΔI_B is always smaller than ΔI , in accordance with the exact inequality (11), and approaches monotonically ΔI as the order B of approximation increases. For small B ,

ΔI_B is significantly smaller than ΔI and attains its minimal value at $B = 0$, corresponding to momentum states ($\Delta I_0 = 0$ in case S). We remark that standard initial states used in previous works are either momentum states or linear superpositions of few momentum states. Thus, the quantum momentum acceleration for the MUSs $|\psi_{\mathbf{w}}\rangle$, measured by ΔI , appears to be the *strongest* quantum ratchet effect known until now.

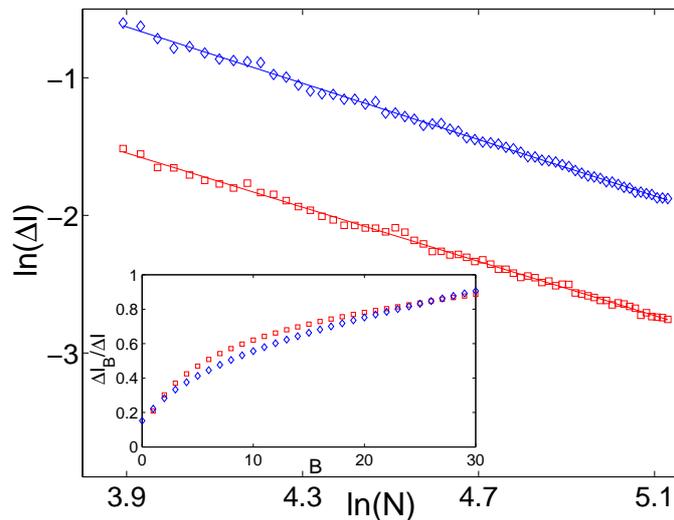


Figure 3. Loglog plots of ΔI versus $N = 2\pi/\hbar$ ($49 \leq N \leq 169$) for $K = 15$ in case S (red squares, with linear fit having slope $\mu = -0.98 \pm 0.01$) and in case A (blue diamonds, $\mu = -1.02 \pm 0.01$). The inset shows $\Delta I_B/\Delta I$ versus B for $N = 121$ and $K = 15$ in the two cases; $\Delta I_{B=0} = 0$ in case S .

3. Conclusion and discussion: open problems

In conclusion, we have presented first results on the quantum ratchet effect in the framework of a new statistical approach. These results concern simple but realistic model systems (the KHMs) in a semiclassical full-chaos regime and for special quantum-resonance values $2\pi/N$ of a scaled Planck constant \hbar . There are several interesting open problems that wait to be analyzed by the new approach. We mention here some of these problems:

(a) To investigate the distribution of the quantum current in a semiclassical regime of mixed phase space, assuming again $\hbar = 2\pi/N$ at least at a first stage. In particular, when accelerator-mode islands are present, the classical chaotic diffusion should be anomalous with a non-Gaussian (approximately Levy) distribution of momenta [31]. This should lead to significant deviations of the quantum momentum current distribution from the estimate (9) (which reflects normal chaotic diffusion), as clearly indicated in Figure 1(a) by the discrepancy of the numerical data from the theoretical prediction (solid line) near $K \approx 6.5$.

(b) To consider generic irrational values of $\hbar/(2\pi)$ for which the KHMs are known [21–23] to exhibit a variety of quantum-transport phenomena depending on the parameters K and L : Ballistic motion, quantum diffusion, and Anderson localization. One would like to understand how precisely these phenomena affect the semiclassical distribution of the quantum current over suitably chosen sets of initial states. Such understanding may be systematically achieved by considering rational approximants q/N of $\hbar/(2\pi)$. When MUSs (2) are used as initial states, one should also examine the dependence of the results on $a = 2\pi q_1/N_1$ and $b = 2\pi q_2/N_2$ for all integers q_1, q_2, N_1 , and N_2 satisfying $q_1 q_2 = q$ and $N_1 N_2 = N$.

(c) To apply the statistical approach to other systems. At a first stage, one would like to consider systems that can be consistently described on a phase-space torus like the KHMs. The quantum ratchet effect in such a system, the well known quantum kicked particle, has been the subject of both theoretical [9, 11] and experimental [16, 17] studies in the framework of the usual approach to the effect. This system is basically different from the quantum KHM since

the quasimomentum $\beta = w_2$ in expressions such as (5)-(7) cannot take all values; the allowed values are determined by a Diophantine equation [32]. This fact should be properly taken into account when investigating the distribution of the quantum current over \mathbf{w} .

Acknowledgments

This work was partially supported by ISF Grant No. 118/05 and BIU Grant No. 2046.

References

- [1] Flach S, Yevtushenko O and Zolotaryuk Y 2000 *Phys. Rev. Lett.* **84** 2358
- [2] Dittrich T, Ketzmerick R, Otto M F and Schanz H 2000 *Ann. Phys. (Leipzig)* **9** 755
- [3] Schanz H, Otto M F, Ketzmerick R and Dittrich T 2001 *Phys. Rev. Lett.* **87** 070601
Schanz H, Dittrich T and Ketzmerick R 2005 *Phys. Rev. E* **71** 026228
- [4] Denisov S and Flach S 2001 *Phys. Rev. E* **64** 056236
Denisov S, Klafter J, Urbakh M and Flach S 2002 *Physica D* **170** 131
Denisov S, Flach S, Ovchinnikov A A, Yevtushenko O and Zolotaryuk Y 2002 *Phys. Rev. E* **66** 041104
- [5] Cheon T, Exner P and Seba P 2003 *J. Phys. Soc. Jpn.* **72** 1087
- [6] Gong J and Brumer P 2004 *Phys. Rev. E* **70** 016202
- [7] Monteiro T S, Dando P A, Hutchings N A C and Isherwood M R 2002 *Phys. Rev. Lett.* **89** 194102
Hutchings N A C, Isherwood M R, Jonckheere T and Monteiro T S 2004 *Phys. Rev. E* **70** 036205
Hur G, Creffield C E, Jones P H and Monteiro T S 2005 *Phys. Rev. A* **72** 013403
- [8] Denisov S, Morales-Molina L, Flach S and Hänggi P 2007 *Phys. Rev. A* **75** 063424
- [9] Lundh E and Wallin M 2005 *Phys. Rev. Lett.* **94** 110603
Kenfack A, Gong J and Pattanayak A K 2008 *Phys. Rev. Lett.* **100** 044104
- [10] Gong J and Brumer P 2006 *Phys. Rev. Lett.* **97** 240602
Pelc J, Gong J and Brumer P 2009 *Phys. Rev. E* **79** 066207
- [11] Dana I and Roitberg V 2007 *Phys. Rev. E* **76** 015201(R)
- [12] Wang J and Gong J 2008 *Phys. Rev. E* **78** 036219
- [13] Dana I 2008 Quantum ratchets on maximally uniform states in phase space: semiclassical full-chaos regime
Preprint nlin.CD/arXiv:0810.1831v1
- [14] Dana I 2010 *Phys. Rev. E* **81** 036210
- [15] Jones P H, Goonasekera M, Meacher D R, Jonckheere T and Monteiro T S 2007 *Phys. Rev. Lett.* **98** 073002
- [16] Sadgrove M, Horikoshi M, Sekimura T and Nakagawa K 2007 *Phys. Rev. Lett.* **99** 043002
- [17] Dana I, Ramareddy V, Talukdar I and Summy G S 2008 *Phys. Rev. Lett.* **100** 024103
Dana I, Roitberg V B, Ramareddy V, Talukdar I and Summy G S 2010 *Int. J. Bifur. Chaos* **20** 255
- [18] Salger T, Kling S, Hecking T, Geckeler C, Morales-Molina L and Weitz M 2009 *Science* **326** 1241
- [19] Hänggi P and Marchesoni F 2009 *Rev. Mod. Phys.* **81** 387
- [20] Leboeuf P, Kurchan J, Feingold M and Arovas D P 1990 *Phys. Rev. Lett.* **65** 3076
Dana I, Feingold M and Wilkinson M 1998 *Phys. Rev. Lett.* **81** 3124
- [21] Lima R and Shepelyansky D 1991 *Phys. Rev. Lett.* **67** 1377
- [22] Geisel T, Ketzmerick R and Petschel P 1991 *Phys. Rev. Lett.* **67** 3635
- [23] Artuso R, Borgonovi F, Guarneri I, Rebuzzini L and Casati G 1992 *Phys. Rev. Lett.* **69** 3302
- [24] Dana I 1994 *Phys. Rev. Lett.* **73** 1609
- [25] Dana I 1995 *Phys. Lett. A* **197** 413
Dana I and Dorofeev D L 2005 *Phys. Rev. E* **72** 046205
- [26] Dana I 1995 *Phys. Rev. E* **52** 466
- [27] Iomin A and Fishman S 1998 *Phys. Rev. Lett.* **81** 1921
Wang J and Gong J 2008 *Phys. Rev. A* **77** 031405(R)
- [28] Duffy G J, Mellish A S, Challis K J and Wilson A C 2004 *Phys. Rev. A* **70** 041602(R)
- [29] Zak J 1968 *Phys. Rev.* **168** 686
- [30] Summy G S 2009 private communication
- [31] Ishizaki R, Horita T, Kobayashi T and Mori H 1991 *Prog. Theor. Phys.* **85** 1013
- [32] Dana I and Dorofeev D L 2006 *Phys. Rev. E* **73** 026206