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General quantization of canonical maps on a two-torus

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Abstract

Canonical maps on a two-torus in phase space are quantized under most general conditions. Recent results by Keating *et al* (1999 *Nonlinearity* **12** 579) are thus fully extended in two directions: (a) The translational component of a general canonical map is included in the quantization. (b) All values of Planck's constant, consistent with the toral boundary conditions (BCs), are considered; generically, these values are rational numbers whose numerator must satisfy a number-theoretical condition. Besides the condition on Planck's constant, the quantization is possible only for particular, 'allowed' BCs on the torus. The general equation determining these BCs is derived. Allowed BCs may *not exist* in some cases; representative examples are the irrational skew translations and Kronecker maps. Exact versions of Egorov's theorem are shown to hold under some conditions. Composition and representation properties of the quantization scheme are studied.

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1. Introduction

Nonintegrable systems whose dynamics can be reduced to a two-torus in phase space have attracted much attention and have become paradigmatic in the field of 'quantum chaos' [1]. Well-known examples are Bloch (crystal) electrons in a magnetic field [2–10], the kicked rotor [11–16], the kicked Harper model [17–28] and the Anosov torus maps (cat maps [29–39] and perturbed cat maps [42–47]). The compactness of a toral phase space is very convenient for studying basic aspects of the classical–quantum correspondence in nonintegrable systems [11–28, 30–40, 42–47] and allows derivation of several exact results, especially for the quantum cat maps [30–40].

Consider in phase space a two-torus $\mathbb{T}^2$ which will be chosen, for simplicity and without loss of generality, as the unit torus (a square of unit area)[1]. In general, a canonical map $\mathbf{\Phi}$ in phase space can be consistently defined on (i.e. taken modulo) $\mathbb{T}^2$ if and only if it can be expressed as the composition of *three* maps (see [45] and section 2): (i) a linear map defined by

[1] An arbitrary torus $\mathbb{T}^2$ can be transformed to the torus $[0, 1) \times [0, 1)$ by a linear transformation of variables.

a matrix in $SL(2, \mathbb{Z})$, (ii) a translation map and (iii) the Poincaré map giving the time-one flow of a Hamiltonian periodic in phase space with unit cell $\mathbb{T}^2$ (the maps (i) and (ii) can be obtained as the time-one flows of nonperiodic Hamiltonians). This composition is *unique*. Once $\boldsymbol{\Phi}$ can be defined on $\mathbb{T}^2$, it can be generally defined on an infinity of other tori 'commensurate' with $\mathbb{T}^2$.

The quantization of several special cases of the general torus map, including the systems mentioned above, have been extensively studied in the quantum-chaos literature. Recently, Keating, Mezzadri and Robbins (KMR) [45] have proposed a natural quantization of canonical torus maps under two restrictions: (a) the maps considered do not contain component (ii) (i.e. zero translations); (b) the simplest values of the (scaled) Planck's constant, consistent with the toral boundary conditions (BCs), were assumed: $h = 1/p$, where p is any integer; this corresponds to defining $\boldsymbol{\Phi}$ on a fixed torus (the unit torus $\mathbb{T}^2$). The main result of KMR is that *not all* BCs are generally allowed. The allowed BCs are determined by an equation involving only component (i) of the classical map.

It is desirable to extend the KMR analysis by removing the restrictions mentioned above. This is for two main reasons. First, the inclusion of a nonvanishing translational component in a torus map usually leads to several interesting phenomena. We mention here, for example, a well-known generalized version of the kicked rotor on the torus [11], corresponding to the composition of the standard map with a translation map; the resulting torus map features all the three basic component maps mentioned above (see section 2). The translational component serves to break in a simple way the time-reversal invariance, thus affecting the spectral statistics [11, 14]. Other examples are the so-called irrational skew translations (ISTs) and Kronecker maps (pure translations on the torus) [36, 37, 40, 41]. These are the simplest torus maps which are fully ergodic (but not chaotic), due entirely to the translational component. The quantization of these maps is interesting for investigating the problem of quantum ergodicity [36, 37, 40].

Secondly, the study of systems such as Bloch electrons in a magnetic field [3–10], the kicked rotor [11–16] and the kicked Harper model [17–28] has shown that it is often convenient and important to assume the most general values of effective Planck's constant for generic torus maps, $h = q/p$, where q and p are coprime integers. The classical and quantum dynamics are then both defined on a torus $\mathbb{T}^2_Q$ (the 'quantum' torus) q times larger than the smallest torus $\mathbb{T}^2$ to which the classical dynamics is reducible. These general values of h are associated with interesting phenomena such as the quantum Hall effect [3–9], quantum resonance [11], quantum antiresonance [16, 24] and tunnelling [21–23], and they have been used to study several properties of the spectral statistics [11, 14] as well as to understand some aspects of the classical–quantum correspondence on a torus [21–23, 26–28]. In addition, the consideration of general rational values of h allows one to approach systematically the generic case of irrational values of h.

In this paper, the quantization of canonical maps on a two-torus is performed under most general conditions. Thus, the two restrictions in the KMR analysis [45] are completely removed. First, the translational component is included and quantized naturally as a Weyl–Heisenberg translation. Secondly, the most general values of $h = q/p$, consistent with the toral BCs, are considered. This is done by exploiting the freedom one has to define the canonical map on particular tori $\mathbb{T}^2_Q$ (called here *admissible* tori), q times larger than $\mathbb{T}^2$. It is shown that, generically, q must be an integer satisfying a number-theoretical condition which involves only component (i) of the map. Besides this condition on Planck's constant, quantization on $\mathbb{T}^2_Q$ is possible only for the allowed BCs on $\mathbb{T}^2_Q$. The equation determining these BCs in the most general case is derived: it involves components (i) and (ii) of the map. Allowed BCs may *not exist* in some cases. Representative examples are precisely the ISTs and Kronecker maps; recent quantization schemes [40, 41] for these maps are considered in the general-BCs framework. An exact version of Egorov's theorem is shown to hold under some conditions. Composition and representation properties of the quantization scheme are studied.

The paper is organized as follows. The expression of the general canonical torus map as the composition of three basic maps is reviewed and illustrated in section 2. In section 3, we determine the conditions to be satisfied by admissible tori $\mathbb{T}_Q^2$ in three main cases. The general quantization on admissible tori is then performed in several stages in section 4. In section 5, we show that exact versions of Egorov's theorem hold under some conditions. Composition and representation properties of the quantization scheme are studied in section 6. In section 7, we illustrate the main concepts and results by some examples, including the ISTs and the Kronecker maps. Conclusions are presented in section 8.

2. The general canonical torus map

We denote by $\boldsymbol{z} = (u, v)$ the phase-space variables (u: position, v: momentum)[2]. In the (u, v) phase space, we consider a torus $\mathbb{T}^2$ that will be chosen, for simplicity and without loss of generality (see footnote 1), as the unit torus $[0, 1) \times [0, 1)$. Let us briefly review some known facts concerning maps on a two-torus (see, e.g., [45] and references therein). The most general canonical (i.e. smooth, area preserving and orientation preserving) map ϕ on $\mathbb{T}^2$ can be written as $\boldsymbol{z} \mapsto \phi(\boldsymbol{z}) = \boldsymbol{\Phi}(\boldsymbol{z}) \bmod 1$. Here the *lifted* map $\boldsymbol{\Phi}(\boldsymbol{z})$, defined on the entire phase plane (u, v), can be expressed uniquely as the composition of three maps (see equation (3.43) in [45])[3]:

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0} \circ \boldsymbol{\Phi}_{\boldsymbol{F}}. \tag{1}$$

The nature of these three maps is as follows. First, $\boldsymbol{\Phi}_A$ is a linear map, $\boldsymbol{\Phi}_A(\boldsymbol{z}) = A \cdot \boldsymbol{z}$, where A is a 2×2 integer matrix with $\det(A) = 1$, i.e. $A \in SL(2, \mathbb{Z})$. Such a map can be obtained as the time-one flow of some quadratic Hamiltonian,

$$H_{\mathrm{q}}(\boldsymbol{z}, t) = \frac{a(t)}{2}u^2 + b(t)uv + \frac{c(t)}{2}v^2 \tag{2}$$

i.e. $\boldsymbol{z}(1) = A \cdot \boldsymbol{z}(0)$, where $\boldsymbol{z}(t)$ is the time evolution of $\boldsymbol{z}$ under (2). In other words [48], we say that (2) is a generator of $\boldsymbol{\Phi}_A$. If $\mathrm{Tr}(A) \geqslant -2$, $\boldsymbol{\Phi}_A$ can be generated by a constant (time-independent) Hamiltonian (2) [33]. The simplest generator of $\boldsymbol{\Phi}_A$ for $\mathrm{Tr}(A) < -2$ is the piecewise constant Hamiltonian defined, for $0 \leqslant t \leqslant 1$, by $H_{\mathrm{q}} = 2\pi(u^2 + v^2)$ for $0 \leqslant t < 0.5$ (this generates the inversion $\boldsymbol{z}(0.5) = -\boldsymbol{z}(0)$) and $H_{\mathrm{q}} = 2H_{\mathrm{q},-A}$ for $0.5 \leqslant t \leqslant 1$, where $H_{\mathrm{q},-A}$ is the constant Hamiltonian generating $\boldsymbol{\Phi}_{-A}$. As for a general canonical map, a given map $\boldsymbol{\Phi}_A$ can be generated by infinitely many different Hamiltonians.

Secondly, the map $\boldsymbol{\Phi}_{\boldsymbol{z}_0}$ is a translation by some vector $\boldsymbol{z}_0$, $\boldsymbol{\Phi}_{\boldsymbol{z}_0}(\boldsymbol{z}) = \boldsymbol{z} + \boldsymbol{z}_0$. This map is generated by linear Hamiltonians, $H_{\mathrm{l}}(\boldsymbol{z}, t) = a(t)u + b(t)v$ (e.g. $a(t) = -v_0$, $b(t) = u_0$). It is convenient and significant to use the notation $\boldsymbol{\Phi}_{A,\boldsymbol{z}_0}$ for the composition $\boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0}$. The map $\boldsymbol{\Phi}_{A,\boldsymbol{z}_0}$ is generated by a Hamiltonian containing both quadratic and linear terms, and one has the relation

$$\boldsymbol{\Phi}_{A,\boldsymbol{z}_0} \equiv \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0} = \boldsymbol{\Phi}_{A \cdot \boldsymbol{z}_0} \circ \boldsymbol{\Phi}_A. \tag{3}$$

[2] All the vectors in this paper should be understood as column vectors. In the text, however, they will be written, for simplicity of notation, as row vectors.

[3] Equation (3.43) in [45] reads, in our notation, $\boldsymbol{\Phi} = \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{F}} \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0}$ ($\boldsymbol{\Phi}_{\boldsymbol{F}}$ and $\boldsymbol{\Phi}_{\boldsymbol{z}_0}$ are denoted, in [45], by $\boldsymbol{\Psi}(1)$ and $\boldsymbol{T}^{\mathrm{cl}}(\boldsymbol{\Delta})$, respectively). For convenience, however, we express the lifted map in the completely equivalent form (1).

Finally, the map Φ_F is defined by $\Phi_F(z) = z + F(z)$, where the vector function $F(z)$ is smooth, periodic with unit cell $\mathbb{T}^2$, and satisfies $\int_{\mathbb{T}^2} F(z)\,\mathrm{d}z = \mathbf{0}$ and $\mathrm{Tr}(DF) + \det(DF) = 0$, DF being the Jacobian matrix of F. These conditions on F are sufficient and necessary for Φ_F to be the time-one flow of a Hamiltonian $H_F(z, t)$ periodic in phase space with unit cell $\mathbb{T}^2$:

$$H_F(z, t) = \sum_{n} H_n(t) \exp(2\pi \mathrm{i} z \wedge n) \tag{4}$$

where $n = (n_1, n_2) \in \mathbb{Z}^2$ is an integer vector and $z \wedge n \equiv un_2 - vn_1$. A famous example is the Harper Hamiltonian $H_F = a\cos(2\pi u) + b\cos(2\pi v)$, arising in the theory of Bloch electrons in a magnetic field [3, 4, 6–10]. Another well-known example is the kicked Harper Hamiltonian $H_F = b\cos(2\pi v) + a\cos(2\pi u)\sum_{s=-\infty}^{\infty} \delta(t - s)$ [17–28]. In this case, the function $F(z)$ for the time-one flow from $t = s - 0$ to $t = s + 1 - 0$ can be explicitly written as $F(z) = 2\pi\{-b\sin[2\pi v + 4\pi^2 a\sin(2\pi u)], a\sin(2\pi u)\}$. It should be emphasized that, in general, $\mathbb{T}^2$ is *not* the basic (smallest) unit cell of periodicity of $F(z)$ or of $H_F(z, t)$, see example below and next section.

As an example of (1), we consider the generalized kicked rotor, defined by the Hamiltonian

$$H = \frac{(v - \lambda)^2}{2} + V(u) \sum_{s=-\infty}^{\infty} \delta(t - s) \tag{5}$$

where λ is a constant and the potential $V(u)$ is an arbitrary smooth periodic function of u with period 1. The Hamiltonian (5), introduced by Izrailev [11] (see also [14]), is usually studied on a cylindrical phase space. Here, however, it is considered on the entire phase plane. The time-one flow for (5), from $t = s - 0$ to $t = s + 1 - 0$, is given by

$$\Phi : \begin{cases} u_{s+1} = u_s + v_{s+1} - \lambda \\ v_{s+1} = v_s - V'(u_s) \end{cases} \tag{6}$$

where $u_s \equiv u(t = s - 0)$, etc. The unique decomposition of (6) into three maps as in (1) can be easily determined. First, Φ_A is associated with the 'parabolic' matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Secondly, the translation map $\Phi_{z_0}(z)$ is associated with the vector $z_0 = (-\lambda, 0)$. Finally,

$$\Phi_F : \begin{cases} u_{s+1} = u_s \\ v_{s+1} = v_s - V'(u_s) \end{cases} \tag{8}$$

so that $F(z) = [0, -V'(u)]$ in this case and, clearly, $\int_{\mathbb{T}^2} F(z)\,\mathrm{d}z = \mathbf{0}$. However, the basic unit cell of $F(z)$ is not $\mathbb{T}^2$ but rather the one-dimensional torus $0 \leqslant u < 1$. The Hamiltonian (5) is invariant under time reversal only for $z_0 = \mathbf{0}$ [11]. In fact, the parameter λ is analogous to a vector potential for a magnetic field in 2D Hamiltonians.

3. Admissible tori

Equation (1) gives the general expression of a canonical map Φ which can be defined on (i.e. taken modulo) a *given* basic torus $\mathbb{T}^2$ (the unit torus). Now, given Φ on $\mathbb{T}^2$, our main question here is under which conditions can Φ be defined on tori different in size from $\mathbb{T}^2$. Such tori will be termed *admissible*.

A general two-torus, to be denoted by $\mathbb{T}^2_Q$, is a parallelogram defined by two vectors R_1 and R_2. These generate a lattice: $R_n = n_1 R_1 + n_2 R_2 = Q \cdot n$, where $n \in \mathbb{Z}^2$ and Q is the nonsingular matrix whose columns are $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$; we shall write $Q = (\boldsymbol{R}_1, \boldsymbol{R}_2)$. The map $\boldsymbol{\Phi}$ can be defined on $\mathbb{T}_Q^2$ (giving a torus map on $\mathbb{T}_Q^2$, $\boldsymbol{z} \mapsto \phi_Q(\boldsymbol{z}) = \boldsymbol{\Phi}(\boldsymbol{z}) \bmod \mathbb{T}_Q^2$) if and only if for any vector $\boldsymbol{R}_{\boldsymbol{n}}$ there exists a vector $\boldsymbol{R}_{\boldsymbol{n}'}$ such that

$$\boldsymbol{\Phi}(\boldsymbol{z} + \boldsymbol{R}_{\boldsymbol{n}}) = \boldsymbol{\Phi}(\boldsymbol{z}) + \boldsymbol{R}_{\boldsymbol{n}'} \tag{9}$$

for all $\boldsymbol{z}$. We then have

Theorem 1. *(i)* $\boldsymbol{R}_{\boldsymbol{n}'} = A \cdot \boldsymbol{R}_{\boldsymbol{n}}$. *(ii)* $\boldsymbol{F}(\boldsymbol{z})$ *is periodic with unit cell* $\mathbb{T}_Q^2$. *(iii) The matrix* $A_Q = Q^{-1}AQ$ *is integer.*

Proof.

(i) Using (1) in (9) we find that

$$\boldsymbol{R}_{\boldsymbol{n}'} = A \cdot [\boldsymbol{R}_{\boldsymbol{n}} + \boldsymbol{F}(\boldsymbol{z} + \boldsymbol{R}_{\boldsymbol{n}}) - \boldsymbol{F}(\boldsymbol{z})]. \tag{10}$$

Integrating equation (10) over $\mathbb{T}^2$ and using the periodicity of $\boldsymbol{F}(\boldsymbol{z})$ on $\mathbb{T}^2$, we obtain $\boldsymbol{R}_{\boldsymbol{n}'} = A \cdot \boldsymbol{R}_{\boldsymbol{n}}$.

(ii) Using $\boldsymbol{R}_{\boldsymbol{n}'} = A \cdot \boldsymbol{R}_{\boldsymbol{n}}$ in (10), we see that $\boldsymbol{F}(\boldsymbol{z} + \boldsymbol{R}_{\boldsymbol{n}}) = \boldsymbol{F}(\boldsymbol{z})$.

(iii) Using $\boldsymbol{R}_{\boldsymbol{n}} = Q \cdot \boldsymbol{n}$ in $\boldsymbol{R}_{\boldsymbol{n}'} = A \cdot \boldsymbol{R}_{\boldsymbol{n}}$, we find that $\boldsymbol{n}' = Q^{-1}AQ \cdot \boldsymbol{n}$ for all $\boldsymbol{n}$, implying that $A_Q = Q^{-1}AQ$ must be integer. □

For a given number $q \neq 0$, we say that A [$\in SL(2, \mathbb{Z})$] is *q-admissible* if there exists a matrix Q with $\det(Q) = q$ such that $Q^{-1}AQ$ is integer. Thus, $\boldsymbol{\Phi}$ can be defined on the (admissible) torus $\mathbb{T}_Q^2$, whose area is $|q|$, if and only if A is q-admissible. From now on, we shall restrict q to be positive so that transformations such as $\boldsymbol{R}_{\boldsymbol{n}} = Q \cdot \boldsymbol{n}$ will be orientation preserving. To obtain more information about admissible tori, we have to make some assumptions concerning the function $\boldsymbol{F}(\boldsymbol{z})$. We thus consider three main cases of $\boldsymbol{F}(\boldsymbol{z})$ in sections 3.1–3.3.

3.1. Case of $\boldsymbol{F}$ *depending on both variables* (u, v)

We shall consider in detail first the generic case that the function $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{F}(u, v)$ depends on both variables (u, v). Then, while $\boldsymbol{F}(\boldsymbol{z})$ must be periodic with unit cell $\mathbb{T}^2$, $\mathbb{T}^2$ is *not*, in general, the basic (smallest) unit cell of $\boldsymbol{F}(\boldsymbol{z})$ but only an integer multiple of it (see an example in section 3.4). For definiteness, however, we shall always assume that the basic unit cell of $\boldsymbol{F}(\boldsymbol{z})$ in this case is $\mathbb{T}^2$. It then follows immediately from theorem 1(ii) that $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ must be vectors in $\mathbb{Z}^2$, so that Q, and thus also $q = \det(Q)$ is *integer*.

It is easy to see that there always exist integers $q > 1$ such that A is q-admissible. In fact, if Q is any integer matrix commuting with A, e.g. an arbitrary polynomial of A with integer coefficients, then $A_Q = A$ and q can assume infinitely many values. An interesting example of $[A, Q] = 0$ arises in the problem studied in [47].

As a simple example of $A_Q \neq A$ ($[A, Q] \neq 0$), consider the case of a q-admissible A with diagonal Q, $Q = \mathrm{diag}(q_1, q_2)$, $q_1q_2 = q$ (i.e. $\mathbb{T}_Q^2$ is the rectangular torus $[0, q_1) \times [0, q_2)$). We easily find that A_Q differs from A only in the off-diagonal elements, given by $A_{Q,1,2} = A_{1,2}q_2/q_1$, $A_{Q,2,1} = A_{2,1}q_1/q_2$. Thus, A is q-admissible with diagonal Q if and only if there exist integers q_1 and q_2, $q_1q_2 = q$, such that q_1 (q_2) divides $A_{1,2}q_2$ ($A_{2,1}q_1$). If q is square-free, i.e. $q = \prod_j p_j$, where p_j are distinct primes, we see that A is q-admissible if and only if q divides $A_{1,2}A_{2,1}$.

In general, the necessary and sufficient condition for the q-admissibility of a matrix $A \in SL(2, \mathbb{Z})$ for integer Q is

Theorem 2 (Z Rudnick). *Let $A \in SL(2,\mathbb{Z})$ and q' be the square-free part of q, i.e. the unique square-free integer q' such that $q = l^2 q'$, l integer. Then A is q-admissible if and only if* $\mathrm{Tr}(A)^2 - 4$ *is a square modulo every odd prime dividing q' and, in addition,* $\mathrm{Tr}(A)$ *is even if q' is even.*

The proof of this theorem is given in the appendix. It follows from the theorem that if A is both q_1- and q_2-admissible, it is also q_1q_2-admissible; thus, it is sufficient to study q-admissibility for prime q. It is instructive to check the theorem in the case considered above (A is q-admissible with diagonal Q). In this case, $A_{1,2}A_{2,1}$ must be a multiple of q if q is square free. Using $\det(A) = 1$, it follows that $\mathrm{Tr}(A)^2 - 4$ is indeed a square modulo every odd prime p dividing q: $\mathrm{Tr}(A)^2 - 4 = (A_{1,1} - A_{2,2})^2 + 4A_{1,2}A_{2,1} = (A_{1,1} - A_{2,2})^2 \bmod p$.

3.2. Case of $\boldsymbol{F}$ depending only on one variable

Consider now the case that $\boldsymbol{F}(\boldsymbol{z})$ depends only on one variable, say u (as in the example in the previous section, see (8)), so that its basic unit cell can be chosen, without loss of generality, as the one-dimensional torus $0 \leqslant u < 1$. The only requirements on Q following from theorem 1 above are now that $Q_{1,1}$ and $Q_{1,2}$ are integers (from theorem 1(ii)) and that $A_Q = Q^{-1}AQ$ is integer. For simplicity, let us restrict ourselves to rectangular tori. Then, by generalizing the case of diagonal Q in section 3.1, we easily find that an admissible torus is defined by the (generally noninteger) matrix $Q = \mathrm{diag}(q_1, q_2)$; here q_1 is an arbitrary nonzero integer and, if $A_{1,2} \neq 0$, $q_2 = rq_1/A_{1,2}$, where r is any integer dividing $A_{1,2}A_{2,1}$. Thus, if $A_{1,2} \neq 0$, the smallest admissible torus is $|A_{1,2}|$ times smaller than $\mathbb{T}^2$. If $A_{1,2} = 0$, $q_2 = A_{2,1}q_1/s$, where s is any nonzero integer, so that an admissible torus can be arbitrarily smaller than $\mathbb{T}^2$. If, in addition, $A_{2,1} = 0$ ($A = I$), an admissible torus can be also one-dimensional, i.e. any integer multiple of the torus $0 \leqslant u < 1$. We thus see that q-admissibility may be possible in this case for (at least some) rational values of q.

3.3. Case of $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$

In this case, $\boldsymbol{\Phi}$ is just the composition $\boldsymbol{\Phi} = \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0} = \boldsymbol{\Phi}_{A,\boldsymbol{z}_0}$, which does not exhibit a natural periodicity in phase space. As a consequence, theorem 1 yields only one requirement on Q: $A_Q = Q^{-1}AQ$ must be integer. Thus, there is much arbitrariness in the choice of Q (its entries may even be irrational), and it is easy to see that q-admissibility is now possible for all values of $q \neq 0$.

3.4. Equivalent map on $\mathbb{T}^2$

One may replace the definition of $\boldsymbol{\Phi}$ on an admissible torus $\mathbb{T}^2_Q$ by the equivalent definition of the map $\boldsymbol{\Phi}^{(Q)}(\boldsymbol{z}) \equiv Q^{-1} \cdot \boldsymbol{\Phi}(\boldsymbol{z}) \cdot Q$ on the fixed unit torus $\mathbb{T}^2$. This replacement is accomplished by the linear transformation of variables $\boldsymbol{z}' = Q^{-1} \cdot \boldsymbol{z}$. It is easy to see that $\boldsymbol{\Phi}^{(Q)} = \boldsymbol{\Phi}_{A_Q} \circ \boldsymbol{\Phi}_{Q^{-1}\cdot\boldsymbol{z}_0} \circ \boldsymbol{\Phi}_{\boldsymbol{F}_Q}$, where $A_Q = Q^{-1}AQ$ and $\boldsymbol{F}_Q(\boldsymbol{z}) = Q^{-1} \cdot \boldsymbol{F}(Q \cdot \boldsymbol{z})$. This transformation, however, usually causes a significant change in the natural periodicity of $\boldsymbol{F}(\boldsymbol{z})$. For example, in the generic case considered in section 3.1, the basic unit cell of $\boldsymbol{F}_Q(\boldsymbol{z})$ is the parallelogram $Q^{-1} \cdot \mathbb{T}^2$ which is q times smaller than $\mathbb{T}^2$. Thus, $\boldsymbol{F}_Q(\boldsymbol{z})$ is not in the class of functions that we assumed in this case and, in addition, the geometry of its basic unit cell is as complicated as that of $\mathbb{T}^2_Q$. In view of this, we shall replace $\boldsymbol{\Phi} \bmod \mathbb{T}^2_Q$ by $\boldsymbol{\Phi}^{(Q)} \bmod \mathbb{T}^2$ only in the case of $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$ (described in section 3.3). This means, in practice, that we can assume $Q = I$ in this case.

4. General quantization on a two-torus

In this section, the quantization of a canonical map $\mathbf{\Phi}$ on a two-torus will be performed under most general conditions. Our quantization scheme is analogous to, but extends the one given in [45] (see the introduction), and it will consist of several stages. First, in section 4.1, we quantize the map $\mathbf{\Phi}$ defined on the entire phase plane (the lifted map). Then, in section 4.2, we consider the boundary conditions (BCs) to be satisfied by proper quantum states on an arbitrary admissible torus $\mathbb{T}^2_Q$ and the resulting condition on Planck's constant. In section 4.3, we review basic facts concerning general toral states using the notions of kq states and kq representation [2]. Finally, in section 4.4, the quantization on $\mathbb{T}^2_Q$ is performed by introducing a general method for defining the quantized lifted map in the spaces of toral states. The equation determining the allowed BCs, for which the quantization is possible, is derived.

4.1. Quantization of the lifted map

The quantum object corresponding to the map $\mathbf{\Phi}$ on the entire phase plane is a unitary evolution operator $\hat{U}$ which normally acts on the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions. In analogy to composition (1), $\hat{U}$ will be naturally expressed as the product of three unitary operators:

$$\hat{U} = \hat{U}_A \hat{U}_{z_0} \hat{U}_{F}. \tag{11}$$

First, the quantization $\hat{U}_A$ of the linear map $\mathbf{\Phi}_A(z) = A \cdot z$ [$A \in SL(2, \mathbb{Z})$] is defined as follows. If a generator of $\mathbf{\Phi}_A$ is some quadratic Hamiltonian $H_{\mathrm{q}}(z, t)$ in (2), then $\hat{U}_A$ is the time-one evolution operator for the corresponding quantum Hamiltonian, $H_{\mathrm{q}}(\hat{z}, t) = [a(t)\hat{u}^2 + b(t)(\hat{u}\hat{v} + \hat{v}\hat{u}) + c(t)\hat{v}^2]/2$, $[\hat{u}, \hat{v}] = \mathrm{i}\hbar$ (i.e. $|\Psi(1)\rangle = \hat{U}_A|\Psi(0)\rangle$, where $|\Psi(t)\rangle$ is the time evolution of a state under the Schrödinger equation for $H_{\mathrm{q}}(\hat{z}, t)$). All the Hamiltonians (2) generating a given map $\mathbf{\Phi}_A$ will generate, after this quantization, the same operator $\hat{U}_A$ up to a sign factor [48]. In the position (u) representation, $\hat{U}_A$ is given, up to a sign factor, by the well-known expression

$$\langle u|\hat{U}_A|u'\rangle = \left(\frac{1}{\mathrm{i}hA_{1,2}}\right)^{1/2} \exp\left[\frac{\mathrm{i}}{2\hbar A_{1,2}}\left(A_{1,1}u'^2 - 2u'u + A_{2,2}u^2\right)\right]. \tag{12}$$

For $A_{1,2} = 0$, (12) reduces to $\langle u|\hat{U}_A|u'\rangle = A_{1,1}^{-1/2}\delta(u - A_{1,1}u') \exp\left[\mathrm{i}A_{1,1}A_{2,1}u^2\big/(2\hbar)\right]$ [48], where, of course, $A_{1,1}$ can assume only the values ± 1. The operators $\pm\hat{U}_A$ form a subgroup of the metaplectic group.

Next, $\hat{U}_{z_0}$, the quantization of the translation map $\mathbf{\Phi}_{z_0}$, is given in a natural way by the Weyl–Heisenberg phase-space translation

$$\hat{U}_{z_0} = \hat{D}(z_0) \equiv \exp\left(\frac{\mathrm{i}}{\hbar}\hat{z} \wedge z_0\right). \tag{13}$$

where $\hat{z} \wedge z_0 \equiv \hat{u}v_0 - \hat{v}u_0$. One has the relations

$$\hat{D}(z)\hat{D}(z') = \exp\left(\frac{\mathrm{i}}{2\hbar}z' \wedge z\right)\hat{D}(z + z') = \exp\left(\frac{\mathrm{i}}{\hbar}z' \wedge z\right)\hat{D}(z')\hat{D}(z). \tag{14}$$

$\hat{U}_{z_0}$ can be interpreted as the time-one evolution operator for some quantum linear Hamiltonian $\hat{H}_1(\hat{z}, t) = a(t)\hat{u} + b(t)\hat{v}$. All the classical linear Hamiltonians generating a given map $\mathbf{\Phi}_{z_0}$ will generate, after quantization, the same operator $\hat{U}_{z_0}$ up to an arbitrary phase factor [48]. Thus, strictly speaking, the quantization of $\mathbf{\Phi}_{z_0}$ can be any of the operators $\exp(2\pi\mathrm{i}\chi)\hat{D}(z_0)$, for arbitrary χ. These operators, for all χ and z_0, form the Weyl–Heisenberg group. For convenience, however, we fix the value of the phase to $\chi = 0$ in (13). The operators $\hat{U}_{A,z_0} \equiv \hat{U}_A\hat{D}(z_0)$ belong to the inhomogeneous metaplectic group, generated by Hamiltonians containing both quadratic and linear terms [48], and satisfy the relation

$$\hat{U}_{A,\boldsymbol{z}_0} \equiv \hat{U}_A \hat{D}(\boldsymbol{z}_0) = \hat{D}(A \cdot \boldsymbol{z}_0)\hat{U}_A \tag{15}$$

which is the quantum analogue of (3).

Finally, the quantization $\hat{U}_{\boldsymbol{F}}$ of $\boldsymbol{\Phi}_{\boldsymbol{F}}$ is defined as follows. Given a periodic Hamiltonian (4) generating $\boldsymbol{\Phi}_{\boldsymbol{F}}$, its Weyl quantization is

$$H_{\boldsymbol{F}}(\hat{\boldsymbol{z}}, t) = \sum_{\boldsymbol{n}} H_{\boldsymbol{n}}(t) \exp(2\pi \mathrm{i} \hat{\boldsymbol{z}} \wedge \boldsymbol{n}) = \sum_{\boldsymbol{n}} H_{\boldsymbol{n}}(t) \hat{D}(\boldsymbol{n}h). \tag{16}$$

Then $\hat{U}_{\boldsymbol{F}}$ is the time-one evolution operator for $H_{\boldsymbol{F}}(\hat{\boldsymbol{z}}, t)$. Clearly, $\hat{U}_{\boldsymbol{F}}$ has the same periodicity in $\hat{\boldsymbol{z}}$ as $H_{\boldsymbol{F}}(\hat{\boldsymbol{z}}, t)$ and can be thus expanded as in (16):

$$\hat{U}_{\boldsymbol{F}} = \sum_{\boldsymbol{n}} U_{\boldsymbol{n}} \exp(2\pi \mathrm{i} \hat{\boldsymbol{z}} \wedge \boldsymbol{n}) = \sum_{\boldsymbol{n}} U_{\boldsymbol{n}} \hat{D}(\boldsymbol{n}h). \tag{17}$$

One can expect that all the Hamiltonians (4) generating a given map $\boldsymbol{\Phi}_{\boldsymbol{F}}$ will generate, after quantization (16), operators (17) that will differ *not* just by a phase factor (compare with the cases above of $\hat{U}_A$ and $\hat{U}_{\boldsymbol{z}_0}$). This is because of the nonlinearity of Hamilton's equations for (4). There will then be many different but equally valid quantizations of $\boldsymbol{\Phi}_{\boldsymbol{F}}$ having the same classical limit, and we shall arbitrarily choose one of them.

4.2. The toral quantum conditions

Consider a general admissible torus $\mathbb{T}^2_Q$, $Q = (\boldsymbol{R}_1, \boldsymbol{R}_2)$. A proper quantum state $|\Psi\rangle$ on the toral phase space $\mathbb{T}^2_Q$ is required to satisfy natural boundary conditions (BCs): $|\Psi\rangle$ must be invariant, up to constant phase factors, under the application of the phase-space translations $\hat{D}(\boldsymbol{R}_j) = \exp(\mathrm{i}\hat{\boldsymbol{z}} \wedge \boldsymbol{R}_j/\hbar)$, $j = 1, 2$:

$$\hat{D}(\boldsymbol{R}_j)|\Psi_{\boldsymbol{w}}\rangle = \exp\left(\frac{\mathrm{i}}{\hbar}\boldsymbol{w} \wedge \boldsymbol{R}_j\right)|\Psi_{\boldsymbol{w}}\rangle \qquad j = 1, 2. \tag{18}$$

In equation (18), $|\Psi\rangle$ has been labelled by a vector $\boldsymbol{w}$ specifying the values of the phase factors. The latter have been expressed in a form completely similar to that of $\hat{D}(\boldsymbol{R}_j)$, $j = 1, 2$ ($\boldsymbol{w}$ 'corresponds' to $\hat{\boldsymbol{z}}$). The nature of the toral states $|\Psi_{\boldsymbol{w}}\rangle$ and the vector $\boldsymbol{w}$ will be considered in some detail in section 4.3. Here we only mention that the states $|\Psi_{\boldsymbol{w}}\rangle$ are, according to equation (18), simultaneous eigenstates of the unitary operators $\hat{D}(\boldsymbol{R}_1)$ and $\hat{D}(\boldsymbol{R}_2)$, and they exist only if $\hat{D}(\boldsymbol{R}_1)$ and $\hat{D}(\boldsymbol{R}_2)$ commute. Using relation (14) and the fact that $\boldsymbol{R}_1 \wedge \boldsymbol{R}_2 = \det(Q) = q$, it follows from $[\hat{D}(\boldsymbol{R}_1), \hat{D}(\boldsymbol{R}_2)] = 0$ that Planck's constant must satisfy the condition

$$h = \frac{q}{p} \tag{19}$$

where p is an arbitrary positive integer. In the generic case considered in section 3.1, q is integer. We can and shall always assume that q and p are coprime in this case. In fact, if the largest common factor of q and p is $d > 1$, there exists always an admissible torus $\mathbb{T}^2_{Q'}$ d times smaller than $\mathbb{T}^2_Q$, i.e. $\det(Q') = q' = q/d$; see lemmas 1 and 3 in the appendix. We then replace $\mathbb{T}^2_Q$ by $\mathbb{T}^2_{Q'}$, having the minimal area among all admissible tori for the given rational value of $h = q/p = q'/p'$, where q' and $p' = p/d$ are now coprime. Interesting cases where q and p are not coprime (and, actually, $d = q$) were considered recently in [47]. In the case considered in section 3.2, q can be rational, for example, $q = rq_1^2/A_{1,2}$ or $q = A_{2,1}q_1^2/s$, where r is any integer dividing $A_{1,2}A_{2,1}$ and s and q_1 are arbitrary nonzero integers. We then choose r and q_1 or s and q_1 so that $p = q/h$ will assume its smallest integer value for the given (rational) value of h. Finally, in the case of $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$, we replace $\boldsymbol{\Phi}$ mod $\mathbb{T}^2_Q$ by $\boldsymbol{\Phi}^{(Q)}$ mod $\mathbb{T}^2$ (see section 3.4), so that (19) is replaced by the standard condition $h = 1/p$ for quantization on the fixed torus $\mathbb{T}^2$.

For future purposes, it is convenient to write the BCs (18) in a more general form. Consider the Weyl–Heisenberg translations $\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})$ on lattice vectors $\boldsymbol{R}_{\boldsymbol{n}} = n_1\boldsymbol{R}_1 + n_2\boldsymbol{R}_2$. One has the relation

$$\hat{D}(\boldsymbol{R}_{\boldsymbol{n}}) = (-1)^{pn_1n_2}\hat{D}^{n_1}(\boldsymbol{R}_1)\hat{D}^{n_2}(\boldsymbol{R}_2) \tag{20}$$

which can easily be derived using the first equality in (14) with (19). If $|\Psi_{\boldsymbol{w}}\rangle$ satisfies (18), we see from (20) that

$$\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})|\Psi_{\boldsymbol{w}}\rangle = (-1)^{pn_1n_2}\exp\left(\frac{\mathrm{i}}{\hbar}\boldsymbol{w}\wedge\boldsymbol{R}_{\boldsymbol{n}}\right)|\Psi_{\boldsymbol{w}}\rangle. \tag{21}$$

The range of variation of $\boldsymbol{w}$ can be restricted to the 'dual' torus of $\mathbb{T}^2_Q$, i.e. the torus $\mathbb{T}^2_{Q/p}$ defined by the vectors $\boldsymbol{W}_j = \boldsymbol{R}_j/p$, $j = 1, 2$. To show this, let us expand $\boldsymbol{w}$ in the basis of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$,

$$\boldsymbol{w} = \theta_1\boldsymbol{W}_1 + \theta_2\boldsymbol{W}_2 = \frac{1}{p}Q\cdot\boldsymbol{\theta}. \tag{22}$$

Using (22) and $\boldsymbol{R}_1\wedge\boldsymbol{R}_2 = q$, equation (18) can be expressed in the form

$$\hat{D}(\boldsymbol{R}_1)|\Psi_{\boldsymbol{w}}\rangle = \exp(-2\pi\mathrm{i}\theta_2)|\Psi_{\boldsymbol{w}}\rangle \qquad \hat{D}(\boldsymbol{R}_2)|\Psi_{\boldsymbol{w}}\rangle = \exp(2\pi\mathrm{i}\theta_1)|\Psi_{\boldsymbol{w}}\rangle \tag{23}$$

from which it is obvious that $\boldsymbol{w}$ in (22) can be restricted to the region $0 \leqslant \theta_1, \theta_2 < 1$, corresponding to $\mathbb{T}^2_{Q/p}$. Thus, $\boldsymbol{w}'$ and $\boldsymbol{w}$ give the same BCs if and only if $\boldsymbol{w}' = \boldsymbol{w} + \boldsymbol{W}_{\boldsymbol{n}}$, where $\boldsymbol{W}_{\boldsymbol{n}}$ is a dual lattice vector, $\boldsymbol{W}_{\boldsymbol{n}} = n_1\boldsymbol{W}_1 + n_2\boldsymbol{W}_2 = p^{-1}Q\cdot\boldsymbol{n}$.

4.3. Kq states and toral states

The nature of the simultaneous eigenstates of two commuting phase-space translations is well understood [2, 4, 17, 26–28, 30, 32, 37, 38, 45, 47]. We summarize this understanding here by considering the general case of phase-space translations $\hat{D}(\boldsymbol{R}_1)$ and $\hat{D}(\boldsymbol{R}_2)$ on an arbitrary admissible torus $\mathbb{T}^2_Q$ for $h = q/p$. We start from the (generally nonadmissible) torus $\mathbb{T}^2_K$, defined by the matrix $K = (\boldsymbol{R}_1, \boldsymbol{R}_2/p)$. An important property of $\mathbb{T}^2_K$ is that its area is precisely h. This implies [2] that $\hat{D}(\boldsymbol{R}_1)$ and $\hat{D}(\boldsymbol{R}_2/p)$ form a complete set of commuting unitary operators. The simultaneous eigenstates of these operators are completely specified, up to constant factors, by the corresponding eigenvalues; the latter can be expressed, in analogy to (18), as $\exp(\mathrm{i}\boldsymbol{w}\wedge\boldsymbol{R}_1/\hbar)$ and $\exp[\mathrm{i}\boldsymbol{w}\wedge(\boldsymbol{R}_2/p)/\hbar]$. We can therefore denote the eigenstates simply by $|\boldsymbol{w}\rangle$. Using arguments similar to those leading to equations (23), one can easily verify that the range of variation of $\boldsymbol{w}$, i.e. the dual torus of $\mathbb{T}^2_K$, is now precisely $\mathbb{T}^2_K$ ($\mathbb{T}^2_K$ is 'self-dual'). The states $|\boldsymbol{w}\rangle$, for all $\boldsymbol{w}\in\mathbb{T}^2_K$, are extended, i.e. they are not normalizable, but are orthonormal in the sense of distributions. For example, in the case of a rectangular torus $\mathbb{T}^2_Q$ with $Q = \mathrm{diag}(q, 1)$ and $K = \mathrm{diag}(q, 1/p)$, the u representation $\langle u|\boldsymbol{w}\rangle$ of $|\boldsymbol{w}\rangle$ is given by a kq quasiperiodic distribution [2]:

$$\langle u|\boldsymbol{w}\rangle = \sqrt{p}\sum_{l=-\infty}^{\infty}\exp(2\pi\mathrm{i}lpw_2)\delta(u - w_1 - lq) \tag{24}$$

where w_1 and w_2 are the components of $\boldsymbol{w}$ in the u and v directions, respectively. The distributions (24) are orthonormal in the following sense:

$$\langle\boldsymbol{w}'|\boldsymbol{w}\rangle = \sum_{r=-\infty}^{\infty}\exp(2\pi\mathrm{i}lpw_2)\delta(w_1' - w_1 - lq)\sum_{l=-\infty}^{\infty}\delta(w_2' - w_2 - s/p).$$

In general, we shall refer to $|\boldsymbol{w}\rangle$ as kq states. Being eigenstates of a complete set of operators, the kq states form a complete set of states [2], so that one can expand in terms of them:

$$|\Psi\rangle = \int_{\mathbb{T}_K^2} \mathrm{d}\boldsymbol{w}' \langle \boldsymbol{w}'|\Psi\rangle |\boldsymbol{w}'\rangle. \tag{25}$$

Here $|\Psi\rangle$ is an arbitrary 'physical' state in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions [37]. In particular, $|\Psi\rangle$ can be a state in $L^2(\mathbb{R})$. The amplitude $\langle \boldsymbol{w}|\Psi\rangle$ in (25) is the kq representation [2] of $|\Psi\rangle$.

The kq states are also eigenstates of $\hat{D}(\boldsymbol{R}_j)$, $j = 1, 2$, since $\hat{D}(\boldsymbol{R}_2) = \hat{D}^p(\boldsymbol{R}_2/p)$. Now, an arbitrary vector $\boldsymbol{w}' \in \mathbb{T}_K^2$ can be expressed uniquely as $\boldsymbol{w}' = m\boldsymbol{R}_1/p + \boldsymbol{w}$, for some $m = 0, \ldots, p-1$ and $\boldsymbol{w} \in \mathbb{T}_{Q/p}^2$, where $\mathbb{T}_{Q/p}^2$ is the dual torus of $\mathbb{T}_Q^2$ (see definitions above). It is easy to see that all the p states $\{|m\boldsymbol{R}_1/p + \boldsymbol{w}\rangle\}_{m=0}^{p-1}$ are eigenstates of $\hat{D}(\boldsymbol{R}_j)$ with eigenvalues $\exp(\mathrm{i}\boldsymbol{w} \wedge \boldsymbol{R}_j/\hbar)$, $j = 1, 2$, independent of m. It follows that a general toral state $|\Psi_{\boldsymbol{w}}\rangle$ in (18) is a linear combination of these p states,

$$|\Psi_{\boldsymbol{w}}\rangle = \sum_{m=0}^{p-1} \psi(m; \boldsymbol{w}) |m\boldsymbol{R}_1/p + \boldsymbol{w}\rangle \tag{26}$$

with arbitrary complex coefficients $\{\psi(m; \boldsymbol{w})\}_{m=0}^{p-1} \in \mathbb{C}^p$. Thus, for given BCs (18), the states $|\Psi_{\boldsymbol{w}}\rangle$ form a p-dimensional subspace $\mathcal{S}_p(\boldsymbol{w})$ of $\mathcal{S}'(\mathbb{R})$. Expansion (26) is a special case of (25) with $\langle \boldsymbol{w}'|\Psi\rangle$ δ-peaked on p 'allowed' locations $\{m\boldsymbol{R}_1/p + \boldsymbol{w}\}_{m=0}^{p-1}$ in $\mathbb{T}_Q^2$. The coefficient $\psi(m; \boldsymbol{w})$ is the probability amplitude of being at one of these locations. In the case of $Q = \mathrm{diag}(q, 1)$, the vectors $\{m\boldsymbol{R}_1/p + \boldsymbol{w}\}_{m=0}^{p-1} = \{(mq/p + w_1, w_2)\}_{m=0}^{p-1}$ are essentially 'position' locations, so that $\psi(m; \boldsymbol{w})$ can be interpreted as the 'position' representation of the toral state $|\Psi_{\boldsymbol{w}}\rangle$.

One can represent $\mathcal{S}_p(\boldsymbol{w})$ in other p-dimensional bases. A general basis is given by $|(m_1\boldsymbol{R}_1 + m_2\boldsymbol{R}_2)/p + \boldsymbol{w}\rangle$, $m_1 = 0, \ldots, p_1 - 1$, $m_2 = 0, \ldots, p_2 - 1$, where p_1 and p_2 are integer factors of p ($p_1 p_2 = p$) and $|\boldsymbol{w}'\rangle$ now denotes the kq states on the torus $\mathbb{T}_K^2$ with $K = (\boldsymbol{R}_1/p_2, \boldsymbol{R}_2/p_1)$. The different representations of $\mathcal{S}_p(\boldsymbol{w})$ are related by unitary transformations in $\mathbb{C}^p$. Consider, for example, the basis $\{|m\boldsymbol{R}_2/p + \boldsymbol{w}\rangle\}_{m=0}^{p-1}$ associated with $K = (\boldsymbol{R}_1/p, \boldsymbol{R}_2)$. In the case of $Q = \mathrm{diag}(q, 1)$, the vectors $\{m\boldsymbol{R}_2/p + \boldsymbol{w}\}_{m=0}^{p-1} = \{(w_1, m/p + w_2)\}_{m=0}^{p-1}$ are essentially 'momentum' locations. As one might expect, the corresponding 'momentum' representation is related to the 'position' representation above by a discrete Fourier transform in $\mathbb{C}^p$.

4.4. Quantization on an admissible torus

The quantization of a canonical map $\boldsymbol{\Phi}$ on an admissible torus $\mathbb{T}_Q^2$ will be based on defining the quantum lifted map $\hat{U}$ in the spaces $\mathcal{S}_p(\boldsymbol{w})$ of toral states. This definition requires extending the action of $\hat{U}$ from $L^2(\mathbb{R})$ to the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions [17, 30, 37, 38]. This extension is straightforward in the case of a phase-space translation (such as $\hat{U}_{\boldsymbol{z}_0}$ in (13)) and of $\hat{U}_{\boldsymbol{F}}$ in (17) (a linear combination of phase-space translations). In the case of $\hat{U}_A$, however, the extension requires special care [30, 38]. A general method for defining the full operator (11) in $\mathcal{S}_p(\boldsymbol{w})$ will be described below. Let us first state the main results in this subsection. The action of (11) on $\mathcal{S}_p(\boldsymbol{w})$ is given by

$$\hat{U}\mathcal{S}_p(\boldsymbol{w}) = \mathcal{S}_p(\boldsymbol{w}') \tag{27}$$

where

$$\boldsymbol{w}' = A \cdot (\boldsymbol{w} + \boldsymbol{z}_0) + \tfrac{1}{2} Q \cdot \boldsymbol{y}_Q \mod \mathbb{T}_{Q/p}^2. \tag{28}$$

Here $\boldsymbol{y}_Q \equiv (A_{Q,1,1}A_{Q,1,2}, A_{Q,2,1}A_{Q,2,2}) \bmod 2$, with $A_Q = Q^{-1}AQ$. If $\boldsymbol{w}' = \boldsymbol{w}$, $\mathcal{S}_p(\boldsymbol{w})$ is *invariant* under $\hat{U}$. The action of $\hat{U}$ can then be *restricted* to the invariant subspace $\mathcal{S}_p(\boldsymbol{w})$ of $\mathcal{S}'(\mathbb{R})$. This restriction, which we denote by $\hat{U}(\boldsymbol{w})$, is precisely the quantization of $\boldsymbol{\Phi}$ on $\mathbb{T}^2_Q$ for the BCs specified by $\boldsymbol{w}$. We thus see that the quantization is possible only if there exist 'allowed' values of $\boldsymbol{w}$ for which $\boldsymbol{w}' = \boldsymbol{w}$ in (28). One should note that if p is even and/or $\boldsymbol{y}_Q = \mathbf{0}$ then $Q \cdot \boldsymbol{y}_Q/2$ is a dual lattice vector and can be removed from the right-hand side of (28). Then, the allowed values of $\boldsymbol{w}$ are just the fixed points of the classical map $\boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0}$ mod $\mathbb{T}^2_{Q/p}$. It is often convenient to express the condition $\boldsymbol{w}' = \boldsymbol{w}$ in terms of $\boldsymbol{\theta}$, defined by relation (22). The general equation determining the allowed BCs will then read as follows:

$$A_Q \cdot \left(\boldsymbol{\theta} + pQ^{-1} \cdot \boldsymbol{z}_0\right) + \frac{p}{2}\boldsymbol{y}_Q = \boldsymbol{\theta} \mod 1. \tag{29}$$

The existence or nonexistence of solutions $\boldsymbol{\theta}$ for (29) in some cases will be studied in section 7.

In order to define $\hat{U}$ in $\mathcal{S}_p(\boldsymbol{w})$ and to prove relations (27) and (28), we start from a general square-integrable state $|\Psi\rangle \in L^2(\mathbb{R})$. Using the fact that an arbitrary vector $\boldsymbol{w}' \in \mathbb{T}^2_K$ can be expressed uniquely as $\boldsymbol{w}' = m\boldsymbol{R}_1/p + \boldsymbol{w}$, for some $m = 0, \ldots, p-1$ and $\boldsymbol{w} \in \mathbb{T}^2_{Q/p}$, expansion (25) for $|\Psi\rangle$ can be expressed as follows:

$$|\Psi\rangle = \int_{\mathbb{T}^2_{Q/p}} \mathrm{d}\boldsymbol{w} |\Psi_{\boldsymbol{w}}\rangle \tag{30}$$

where $|\Psi_{\boldsymbol{w}}\rangle$ is given by (26) with

$$\psi(m; \boldsymbol{w}) = \langle m\boldsymbol{R}_1/p + \boldsymbol{w}|\Psi\rangle. \tag{31}$$

Relation (30) can be inverted,

$$|\Psi_{\boldsymbol{w}}\rangle = \hat{P}(\boldsymbol{w})|\Psi\rangle \tag{32}$$

where

$$\hat{P}(\boldsymbol{w}) = \frac{p}{q}\sum_{\boldsymbol{n}} \exp\left(-\frac{\mathrm{i}}{\hbar}\boldsymbol{w} \wedge \boldsymbol{R}_{\boldsymbol{n}}\right)(-1)^{pn_1n_2}\hat{D}(\boldsymbol{R}_{\boldsymbol{n}}) \tag{33}$$

is the projection operator [37] for $\mathcal{S}_p(\boldsymbol{w})$. Relation (32) with (33) is easily derived from (30) using (21) and Poisson's summation formula

$$\sum_{\boldsymbol{n}} \exp\left[\frac{\mathrm{i}}{\hbar}(\boldsymbol{w}' - \boldsymbol{w}) \wedge \boldsymbol{R}_{\boldsymbol{n}}\right] = \sum_{\boldsymbol{n}} \delta(\theta_1' - \theta_1 - n_1)\delta(\theta_2' - \theta_2 - n_2)$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is related to $\boldsymbol{w}$ by (22). It is then clear from (31) that an arbitrary state (26) in $\mathcal{S}_p(\boldsymbol{w})$ can be obtained by applying the projection operator (33) to a properly chosen state $|\Psi\rangle$ in $L^2(\mathbb{R})$.

With this in mind, we can now define the action of $\hat{U}$ on $\mathcal{S}_p(\boldsymbol{w})$ in a natural way on the basis of (32):

$$\hat{U}|\Psi_{\boldsymbol{w}}\rangle \equiv \frac{p}{q}\sum_{\boldsymbol{n}} \exp\left(-\frac{\mathrm{i}}{\hbar}\boldsymbol{w} \wedge \boldsymbol{R}_{\boldsymbol{n}}\right)(-1)^{pn_1n_2}\hat{U}\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})|\Psi\rangle. \tag{34}$$

Let us show that the expression on the right-hand side of (34) is well defined and gives a toral state. First, the states $\hat{U}\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})|\Psi\rangle \in L^2(\mathbb{R})$ and one has

$$\hat{U}\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})|\Psi\rangle = \exp\left(-\frac{\mathrm{i}}{\hbar}\boldsymbol{z}_0 \wedge \boldsymbol{R}_{\boldsymbol{n}}\right)\hat{D}(A \cdot \boldsymbol{R}_{\boldsymbol{n}})\hat{U}|\Psi\rangle. \tag{35}$$

Relation (35) is derived using (11), (14), (15), and the fact that

$$[\hat{U}_{\boldsymbol{F}}, \hat{D}(\boldsymbol{R}_{\boldsymbol{n}})] = 0$$

since $\hat{U}_{\boldsymbol{F}}$ in (17) is periodic on $\mathbb{T}^2$. Next, the lattice $\boldsymbol{R}_{\boldsymbol{n}}$ in (35) is invariant under A, $A \cdot \boldsymbol{R}_{\boldsymbol{n}} = \boldsymbol{R}_{\boldsymbol{n}'}$, where $\boldsymbol{n}' = A_Q \cdot \boldsymbol{n}\ \big(A_Q = Q^{-1}AQ\big)$. One has

$$(-1)^{pn_1n_2} = (-1)^{pn_1'n_2' + p\boldsymbol{y}_Q\wedge\boldsymbol{n}'} \tag{36}$$

where $\boldsymbol{y}_Q$ was defined after equation (28). Relation (36) can be verified as follows. Denoting the integer matrix A_Q simply by $A_Q = (a, b; c, d)$ and using $ad - bc = 1$, we find that

$$\begin{aligned} n_1n_2 - n_1'n_2' &= (ad + bc - 1)n_1'n_2' - abn_2'^2 - cdn_1'^2 \\ &= abn_2' - cdn_1' \bmod 2 = \boldsymbol{y}_Q \wedge \boldsymbol{n}' \bmod 2 \end{aligned}$$

which proves (36). We can write

$$(-1)^{p\boldsymbol{y}_Q\wedge\boldsymbol{n}'} = \exp\Big[-\frac{\mathrm{i}}{\hbar}(Q\cdot\boldsymbol{y}_Q/2)\wedge\boldsymbol{R}_{\boldsymbol{n}'}\Big] \tag{37}$$

by using the relation

$$\boldsymbol{y}_Q \wedge \boldsymbol{n}' = \boldsymbol{y}_Q \wedge \big(Q^{-1}\cdot\boldsymbol{R}_{\boldsymbol{n}'}\big) = \frac{1}{q}(Q\cdot\boldsymbol{y}_Q)\wedge\boldsymbol{R}_{\boldsymbol{n}'}$$

which follows from the identity $\boldsymbol{a}\wedge(Q^{-1}\cdot\boldsymbol{b}) = \det(Q^{-1})(Q\cdot\boldsymbol{a})\wedge\boldsymbol{b}$. Finally, the last identity implies that

$$\boldsymbol{a}\wedge\boldsymbol{R}_{\boldsymbol{n}} = \boldsymbol{a}\wedge(A^{-1}\cdot\boldsymbol{R}_{\boldsymbol{n}'}) = (A\cdot\boldsymbol{a})\wedge\boldsymbol{R}_{\boldsymbol{n}'} \tag{38}$$

for arbitrary $\boldsymbol{a}$. Using (35)–(38), relation (34) can be written as follows:

$$\hat{U}|\Psi_{\boldsymbol{w}}\rangle \equiv \frac{p}{q}\sum_{\boldsymbol{n}}\exp\Big(-\frac{\mathrm{i}}{\hbar}\boldsymbol{w}'\wedge\boldsymbol{R}_{\boldsymbol{n}}\Big)(-1)^{pn_1n_2}\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})\hat{U}|\Psi\rangle = \hat{P}(\boldsymbol{w}')\hat{U}|\Psi\rangle \tag{39}$$

where $\boldsymbol{w}'$ is given by (28) and $\hat{P}(\boldsymbol{w}')$ is the corresponding projection operator. The action of $\hat{U}$ on $|\Psi_{\boldsymbol{w}}\rangle$ is then well defined by relation (39): $\hat{U}|\Psi_{\boldsymbol{w}}\rangle = |\Psi'_{\boldsymbol{w}'}\rangle \in \mathcal{S}_p(\boldsymbol{w}')$, where $|\Psi'_{\boldsymbol{w}'}\rangle$ is obtained by projecting the square-integrable state $|\Psi'\rangle = \hat{U}|\Psi\rangle$ on the space $\mathcal{S}_p(\boldsymbol{w}')$. This definition of $\hat{U}|\Psi_{\boldsymbol{w}}\rangle$ is unambiguous: if $\hat{P}(\boldsymbol{w})|\Psi_1\rangle = \hat{P}(\boldsymbol{w})|\Psi_2\rangle$ for $|\Psi_1\rangle \neq |\Psi_2\rangle$, also $\hat{P}(\boldsymbol{w}')\hat{U}|\Psi_1\rangle = \hat{P}(\boldsymbol{w}')\hat{U}|\Psi_2\rangle$; this is simply because $\hat{P}(\boldsymbol{w})|\Psi\rangle = 0 \Longrightarrow \hat{P}(\boldsymbol{w}')\hat{U}|\Psi\rangle = \hat{U}\hat{P}(\boldsymbol{w})|\Psi\rangle = 0$. It is clear from (39) that any state $\hat{P}(\boldsymbol{w}')|\Psi'\rangle \in \mathcal{S}_p(\boldsymbol{w}')$ can be written as $\hat{U}|\Psi_{\boldsymbol{w}}\rangle$, where $|\Psi_{\boldsymbol{w}}\rangle = \hat{P}(\boldsymbol{w})\hat{U}^{-1}|\Psi'\rangle \in \mathcal{S}_p(\boldsymbol{w})$. This completes the proof of relations (27) and (28).

Let us assume the existence of 'allowed' values of $\boldsymbol{w}$, for which $\boldsymbol{w}' = \boldsymbol{w}$ in (28). Then, as already mentioned above, the action (39) of $\hat{U}$ can be restricted to $\mathcal{S}_p(\boldsymbol{w})$, and the restriction $\hat{U}(\boldsymbol{w})$ is the quantization of $\boldsymbol{\Phi}$ on $\mathbb{T}^2_Q$ for the corresponding BCs. It is natural to represent $\hat{U}(\boldsymbol{w})$ in some p-dimensional basis of $\mathcal{S}_p(\boldsymbol{w})$, say $\{|m\boldsymbol{R}_1/p + \boldsymbol{w}\rangle\}_{m=0}^{p-1}$:

$$\hat{U}(\boldsymbol{w})|m\boldsymbol{R}_1/p + \boldsymbol{w}\rangle = \sum_{m'=0}^{p-1} U_{m',m}(\boldsymbol{w})|m'\boldsymbol{R}_1/p + \boldsymbol{w}\rangle. \tag{40}$$

Since $\hat{U}$ is unitary, the $p \times p$ matrix $U(\boldsymbol{w})$ in (40) is unitary. This matrix has been explicitly calculated in several special cases. See, for example, [26] for the case of the kicked Harper model ($\hat{U} = \hat{U}_{\boldsymbol{F}}$) and [30, 38] for the case of $\hat{U} = \hat{U}_A$. In general, since equation (28) does not depend on $\boldsymbol{F}$, *all* $\boldsymbol{w}$ are allowed for $\hat{U} = \hat{U}_{\boldsymbol{F}}$. The allowed values of $\boldsymbol{w}$ are just those corresponding to the inhomogeneous metaplectic operator $\hat{U}_{A,\boldsymbol{z}_0} = \hat{U}_A\hat{U}_{\boldsymbol{z}_0}$. The composition $\hat{U} = \hat{U}_{A,\boldsymbol{z}_0}\hat{U}_{\boldsymbol{F}}$ implies then a similar composition for the matrix representation $U(\boldsymbol{w})$ in (40):

$$U(\boldsymbol{w}) = U_{A,\boldsymbol{z}_0}(\boldsymbol{w})U_{\boldsymbol{F}}(\boldsymbol{w}). \tag{41}$$

The representation (41) will be considered in more detail in section 6.

5. Exact versions of Egorov's theorem

Having defined the quantum map $\hat{U}(\boldsymbol{w})$ on an admissible torus $\mathbb{T}^2_Q$, we now consider the quantization of a classical observable on $\mathbb{T}^2_Q$ and its dynamical evolution under $\hat{U}(\boldsymbol{w})$. A classical toral observable is described by a smooth periodic function $f(\boldsymbol{z})$ on $\mathbb{T}^2_Q$, with Fourier expansion

$$f(\boldsymbol{z}) = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \exp[2\pi \mathrm{i}(Q^{-1} \cdot \boldsymbol{z}) \wedge \boldsymbol{n}]. \tag{42}$$

The quantum observable corresponding to $f(\boldsymbol{z})$ on the entire phase plane $\mathbb{R}^2$ will be given, for $h = q/p$, by the Weyl quantization of (42):

$$f(\hat{\boldsymbol{z}}) = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \exp[2\pi \mathrm{i}(Q^{-1} \cdot \hat{\boldsymbol{z}}) \wedge \boldsymbol{n})] = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \hat{D}(p^{-1} Q \cdot \boldsymbol{n}). \tag{43}$$

It is easy to check, using (14), that $[f(\hat{\boldsymbol{z}}), \hat{D}(\boldsymbol{R}_{\boldsymbol{n}})] = 0$, where $\hat{D}(\boldsymbol{R}_{\boldsymbol{n}})$ is defined by (20) and satisfies (21). Thus, $f(\hat{\boldsymbol{z}})\mathcal{S}_p(\boldsymbol{w}) = \mathcal{S}_p(\boldsymbol{w})$ for all $\boldsymbol{w}$, so that we can restrict $f(\hat{\boldsymbol{z}})$ to an arbitrary space $\mathcal{S}_p(\boldsymbol{w})$. This restriction, to be denoted by $\mathrm{Op}_{\boldsymbol{w}}(f)$, defines the quantum toral observable corresponding to $f(\boldsymbol{z})$ on $\mathbb{T}^2_Q$ for given BCs (18). One expects that in the classical limit of $p \to \infty$ (with p coprime to q), the evolution of $\mathrm{Op}_{\boldsymbol{w}}(f)$ under $\hat{U}(\boldsymbol{w})$ for allowed $\boldsymbol{w}$ will be given by the quantization of the classical evolution of $f(\boldsymbol{z})$ under the torus map ϕ (Egorov's theorem):

$$\lim_{p \longrightarrow \infty} \left[\hat{U}^{-1}(\boldsymbol{w})\mathrm{Op}_{\boldsymbol{w}}(f)\hat{U}(\boldsymbol{w}) - \mathrm{Op}_{\boldsymbol{w}}(f \circ \phi)\right] = 0. \tag{44}$$

We now show that under some conditions exact versions of (44) hold for any p:

Theorem 3.

(i) *For all torus maps $\phi = \phi_0$ with $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$ (i.e. $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{A,\boldsymbol{z}_0} = \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0}$) and for allowed $\boldsymbol{w}$, one has*

$$\hat{U}^{-1}(\boldsymbol{w})\mathrm{Op}_{\boldsymbol{w}}(f)\hat{U}(\boldsymbol{w}) = \mathrm{Op}_{\boldsymbol{w}}(f \circ \phi_0). \tag{45}$$

(ii) *Defining $\phi_0 \equiv \boldsymbol{\Phi}_{A,\boldsymbol{z}_0} \bmod \mathbb{T}^2_Q$ for generic ϕ, i.e. $\boldsymbol{F}(\boldsymbol{z}) \neq \boldsymbol{0}$, relation (45) is valid if and only if $f(\boldsymbol{z})$ is periodic in $\mathbb{T}^2_Q$ with unit cell $\mathbb{T}^2_{Q/p}$ (the dual torus).*

Proof.

(i) Using (14), (15) and (43), we immediately get, for $\hat{U} = \hat{U}_A \hat{U}_{\boldsymbol{z}_0}$,

$$\hat{U}^{-1} f(\hat{\boldsymbol{z}})\hat{U} = \sum_{\boldsymbol{n}} f_{\boldsymbol{n}} \exp\left\{2\pi \mathrm{i}\left[Q^{-1}A \cdot (\hat{\boldsymbol{z}} + \boldsymbol{z}_0)\right] \wedge \boldsymbol{n}\right\}. \tag{46}$$

But $f[A \cdot (\boldsymbol{z} + \boldsymbol{z}_0)] = f[\boldsymbol{\Phi}_{A,\boldsymbol{z}_0}(\boldsymbol{z})]$, so that $\hat{U}^{-1} f(\hat{\boldsymbol{z}})\hat{U} = f[\boldsymbol{\Phi}_{A,\boldsymbol{z}_0}(\hat{\boldsymbol{z}})]$. By restricting the last relation to $\mathcal{S}_p(\boldsymbol{w})$, we obtain (45).

(ii) Since relation (46) holds for $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$, relation (45) will be valid for generic ϕ if and only if $f(\hat{\boldsymbol{z}})$ commutes with $\hat{U}_{\boldsymbol{F}}$. Using (17), (43) and (14), we get

$$f(\hat{\boldsymbol{z}})\hat{U}_{\boldsymbol{F}} = \sum_{\boldsymbol{n},\boldsymbol{n}'} f_{\boldsymbol{n}} U_{\boldsymbol{n}'} \exp\left[\frac{2\pi \mathrm{i}}{p} \boldsymbol{n}' \wedge (Q \cdot \boldsymbol{n})\right] \hat{D}(\boldsymbol{n}'h)\hat{D}(p^{-1}Q \cdot \boldsymbol{n}). \tag{47}$$

It is then clear that $[f(\hat{\boldsymbol{z}}), \hat{U}_{\boldsymbol{F}}] = 0$ for generic $\hat{U}_{\boldsymbol{F}}$ if and only if the phase factor in (47) is equal to 1 for all $\boldsymbol{n}'$. This implies that $\boldsymbol{n}$ must be a multiple of p, so that $f(\boldsymbol{z})$ in (42) is periodic in $\mathbb{T}^2_Q$ with unit cell $\mathbb{T}^2_{Q/p}$. □

Part (i) of theorem 3 is already known for $\boldsymbol{z}_0 = \boldsymbol{0}$ (see lemma 6.2 in [37]) and for the skew translations [40]. Part (ii) means that if $f(\boldsymbol{z})$ is periodic in the dual torus $\mathbb{T}^2_{Q/p}$ the quantum evolution of the corresponding toral observable is not affected by $\hat{U}_{\boldsymbol{F}}$. This is completely analogous to the 'evolution' (28) of $\boldsymbol{w}$ on $\mathbb{T}^2_{Q/p}$.

6. Composition and representation properties

We study here in some detail the composition and representation properties of the toral quantization $\hat{U}(\boldsymbol{w})$. Consider first the composition of classical maps (1). Using the notation in (3), we write two such maps as $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}_{A,\boldsymbol{z}_1} \circ \boldsymbol{\Phi}_{\boldsymbol{F}}$ and $\boldsymbol{\Phi}_2 = \boldsymbol{\Phi}_{B,\boldsymbol{z}_2} \circ \boldsymbol{\Phi}_{\boldsymbol{G}}$. It is easy to check that

$$\boldsymbol{\Phi}_3 \equiv \boldsymbol{\Phi}_1 \circ \boldsymbol{\Phi}_2 = \boldsymbol{\Phi}_{C,\boldsymbol{z}_3} \circ \boldsymbol{\Phi}_{\boldsymbol{E}} \tag{48}$$

where

$$C = AB \qquad \boldsymbol{z}_3 = \boldsymbol{z}_2 + B^{-1} \cdot \boldsymbol{z}_1 \tag{49}$$

and

$$\boldsymbol{\Phi}_{\boldsymbol{E}} = \boldsymbol{\Phi}_{\tilde{\boldsymbol{F}}} \circ \boldsymbol{\Phi}_{\boldsymbol{G}} \qquad \tilde{\boldsymbol{F}}(\boldsymbol{z}) = B^{-1} \cdot \boldsymbol{F}[B \cdot (\boldsymbol{z} + \boldsymbol{z}_2)]. \tag{50}$$

If $H_{\tilde{\boldsymbol{F}}}(\boldsymbol{z}, t)$ and $H_{\boldsymbol{G}}(\boldsymbol{z}, t)$ are periodic Hamiltonians (4) generating the maps $\boldsymbol{\Phi}_{\tilde{\boldsymbol{F}}}$ and $\boldsymbol{\Phi}_{\boldsymbol{G}}$, respectively, the map $\boldsymbol{\Phi}_{\boldsymbol{E}}$ in (50) can be generated by a Hamiltonian $H_{\boldsymbol{E}}(\boldsymbol{z}, t)$ defined, for $0 \leqslant t \leqslant 1$, by

$$H_{\boldsymbol{E}}(\boldsymbol{z}, t) = \begin{cases} 2H_{\boldsymbol{G}}(\boldsymbol{z}, 2t) & 0 \leqslant t < 0.5 \\ 2H_{\tilde{\boldsymbol{F}}}(\boldsymbol{z}, 2t-1) & 0.5 \leqslant t \leqslant 1. \end{cases} \tag{51}$$

Next, we consider the composition of the corresponding operators (11). Using relations (13)–(17) and the fact that $\hat{U}_A \hat{U}_B = \pm \hat{U}_{AB}$ [48], a straightforward calculation gives the quantum analogue of (48):

$$\hat{U}_3 \equiv \hat{U}_1 \hat{U}_2 = \pm \exp\left[\frac{\mathrm{i}}{2\hbar}(B \cdot \boldsymbol{z}_2) \wedge \boldsymbol{z}_1\right] \hat{U}_{C,\boldsymbol{z}_3} \hat{U}_{\boldsymbol{E}} \tag{52}$$

where $\hat{U}_{\boldsymbol{E}}$ is the time-one evolution operator for the Weyl quantization $H_{\boldsymbol{E}}(\hat{\boldsymbol{z}}, t)$ of (51). Relation (52) shows that the composition law of (11) is, up to a constant phase factor, the same as that of the corresponding classical maps (1). The operators (11) thus provide a satisfactory representation and quantization of (1). In particular, it follows from (52) and (49) that, for any integer $n > 1$,

$$\hat{U}^n = \pm \exp\left[\frac{\mathrm{i}}{2\hbar}\left(\sum_{j=1}^{n-1} j A^{n-j} \cdot \boldsymbol{z}_0\right) \wedge \boldsymbol{z}_0\right] \mathrm{Op}(\boldsymbol{\Phi}^n) \tag{53}$$

where $\mathrm{Op}(\boldsymbol{\Phi}^n)$ is the operator corresponding to the nth iterate of $\boldsymbol{\Phi}$.

Let us now assume that $\hat{U}_1 \mathcal{S}_p(\boldsymbol{w}) = \hat{U}_2 \mathcal{S}_p(\boldsymbol{w}) = \mathcal{S}_p(\boldsymbol{w})$. Then also $\hat{U}_1 \hat{U}_2 \mathcal{S}_p(\boldsymbol{w}) = \mathcal{S}_p(\boldsymbol{w})$. This implies that all classical maps $\boldsymbol{\Phi}_{A,\boldsymbol{z}_0}$ for which $\boldsymbol{w}' = \boldsymbol{w}$ in (28) form a group $\mathcal{G}(\boldsymbol{w})$. Now, relation (39) defines $\hat{U}|\Psi_{\boldsymbol{w}}\rangle$ on the basis of the action of $\hat{U}$ on a state $|\Psi\rangle$ in $L^2(\mathbb{R})$, where the composition law (52) holds. It follows then from relations (39)–(41) that the composition law (52) (in particular, (53)) is inherited by the corresponding toral quantizations and their matrix representations (41):

$$U_3(\boldsymbol{w}) \equiv U_1(\boldsymbol{w}) U_2(\boldsymbol{w}) = \pm \exp\left[\frac{\mathrm{i}}{2\hbar}(B \cdot \boldsymbol{z}_2) \wedge \boldsymbol{z}_1\right] U_{C,\boldsymbol{z}_3}(\boldsymbol{w}) U_{\boldsymbol{E}}(\boldsymbol{w})$$

where $\boldsymbol{\Phi}_{A,\boldsymbol{z}_1}, \boldsymbol{\Phi}_{B,\boldsymbol{z}_2} \in \mathcal{G}(\boldsymbol{w})$.

7. Examples

We illustrate here by some examples the main concepts and results of the previous sections. Consider the generic case described in section 3.1. In this case, the matrix Q is integer and A is q-admissible for $\mathrm{Tr}(A) \neq 2$ only for integers q specified by theorem 2. For example, if $\mathrm{Tr}(A) = 0$, A is q-admissible if and only if the square-free part of q is not divisible by primes p having the property that -4 is not a square modulo p. By quadratic reciprocity, these are the primes $p = 3 \bmod 4$. If $\mathrm{Tr}(A) = 3$, A is q-admissible if and only if the square-free part of q is odd and divisible by 5 or by primes $p = 1, 4 \bmod 5$.

Allowed BCs always exist for $\mathrm{Tr}(A) \neq 2$: equation (29) has the nonempty discrete set of solutions

$$\boldsymbol{\theta}_{\boldsymbol{n}} = (A_Q - I)^{-1} \left(\boldsymbol{n} - pQ^{-1}A \cdot \boldsymbol{z}_0 - \frac{p}{2}\boldsymbol{y}_Q \right) \bmod 1 \qquad \boldsymbol{n} \in \mathbb{Z}^2 \tag{54}$$

The number of distinct solutions (54) is always finite and equal to $/\mathrm{Tr}(A) - 2|$, as the number of fixed points of $\boldsymbol{\Phi}_A \bmod \mathbb{T}^2$, see [33].

Theorem 2 implies that A is q-admissible for all q if $\mathrm{Tr}(A) = 2$. In this case, A can always be written, without loss of generality, in the form

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \tag{55}$$

for some integer k. If $A = I$ ($k = 0$) and $\boldsymbol{z}_0 = \boldsymbol{0}$, *all* $\boldsymbol{\theta}$ are allowed. An example is the kicked Harper model (see section 2). As a second example of (55), let us consider the generalized kicked-rotor map (6), for which $k = 1$ (see (7)) and $\boldsymbol{z}_0 = (-\lambda, 0)$. From the results in section 3.2 it follows that a rectangular admissible torus is defined by the integer matrix $Q = \mathrm{diag}(q_1, q_2)$, where $q_1 q_2 = q$ and $r = q_2/q_1$ is integer. Equation (29) always has a nonempty set of solutions:

$$\theta_1 \text{ arbitrary} \quad (0 \leqslant \theta_1 < 1)$$
$$\theta_2 = l/r + p\lambda/q_2 + 1/2 \quad \mathrm{mod} 1 \qquad l = 0, \ldots, r-1$$

(the term $1/2$ in the equation for θ_2 is removed if pr is even). Thus, the allowed BCs correspond to r equidistant unit segments parallel to the θ_1 axis.

As another example of (55), we consider the case of the skew translations [37, 40], whose lifted map is given by $\boldsymbol{\Phi} = \boldsymbol{\Phi}_{\boldsymbol{z}_0'} \circ \boldsymbol{\Phi}_A$, where $\boldsymbol{z}_0' = (0, \lambda)$ and k in (55) is an arbitrary nonzero integer; writing $\boldsymbol{\Phi}$ in the form of (1), we have $\boldsymbol{\Phi} = \boldsymbol{\Phi}_A \circ \boldsymbol{\Phi}_{\boldsymbol{z}_0}$, where $\boldsymbol{z}_0 = (-k\lambda, \lambda)$. The 'irrational' skew translation (IST), characterized by an irrational value of λ, is known to be a *uniquely* ergodic map on $\mathbb{T}^2$ [49]. This means that there exists only one invariant probability measure (the Lebesgue measure) for the IST. Since $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$ in this case, we can restrict ourselves to the fixed torus $\mathbb{T}^2$, i.e. $Q = I$ (see section 3.4). It is easy to see that equation (29) has a nonempty set of solutions if and only if

$$\lambda = \frac{r}{p} \tag{56}$$

where r is some integer. The solutions are then given by

$$\begin{aligned} &\theta_1 \text{ arbitrary} \quad (0 \leqslant \theta_1 < 1) \\ &\theta_2 = l/k + 1/2 \quad \mathrm{mod} 1 \qquad l = 0, \ldots, k-1 \end{aligned} \tag{57}$$

(the term $1/2$ in the equation for θ_2 is removed if pk is even). If λ is irrational, condition (56) is not satisfied and there exist *no* allowed BCs. Nevertheless, condition (56) may be used to explain in a simple way a quantization procedure for the ISTs proposed recently by Marklof and Rudnick [40] for $k = 2$ and strictly periodic BCs ($\boldsymbol{\theta} = \boldsymbol{0}$). It is clear from (57) that

$\boldsymbol{\theta} = \mathbf{0}$ is indeed an allowed solution if $k = 2$. Now, for irrational λ, choose $r = r(p)$ so as to minimize $\eta = |\lambda - r/p|$ ($\eta_{\min} < 1/p$), and consider the toral quantizations $\hat{U}_p(\mathbf{0})$ of the rational skew translations with $\lambda = \lambda_p = r(p)/p$ for $\boldsymbol{\theta} = \mathbf{0}$. Then $\hat{U}_\lambda \equiv \lim_{p\to\infty} \hat{U}_p(\mathbf{0})$ defines a quantization of the IST with the given irrational value of λ in the classical limit $p \to \infty$. Using an exact version of Egorov's theorem (this is essentially theorem 3(i) in section 5 in the case of skew translations satisfying (56)), Marklof and Rudnick [40] have shown that, in analogy to the classical IST, $\hat{U}_\lambda$ is *uniquely* ergodic quantally: for *all* eigenstates of $\hat{U}_\lambda$, the expectation values of quantum observables coincide with the classical phase-space average (with respect to Lebesgue measure) of the corresponding classical observables.

As a final example, we consider the simple case of the pure translations (Kronecker maps), whose lifted map is just $\boldsymbol{\Phi}_{\boldsymbol{z}_0}$ with arbitrary $\boldsymbol{z}_0 = (\lambda_1, \lambda_2)$. One can show [50] that the Kronecker map is uniquely ergodic on $\mathbb{T}^2$ if and only if $1, \lambda_1, \lambda_2$ are linearly independent over the integers (then λ_1 and λ_2 are necessarily irrational numbers). Since $\boldsymbol{F}(\boldsymbol{z}) = \mathbf{0}$, we can again restrict ourselves to the fixed torus $\mathbb{T}^2$ ($Q = I$). We find that equation (29) has solutions if and only if

$$\lambda_1 = \frac{r_1}{p} \qquad \lambda_2 = \frac{r_2}{p} \tag{58}$$

where r_1 and r_2 are integers. In fact, when (58) is satisfied *all* $\boldsymbol{\theta}$ are allowed. On the other hand, if the Kronecker map is uniquely ergodic (λ_1 and λ_2 are irrational numbers), there exist *no* allowed BCs. The quantization of this map in the classical limit $p \to \infty$ may be performed in a way completely analogous to the quantization procedure described above for the ISTs. Choose first $r_1 = r_1(p)$ and $r_2 = r_2(p)$ so as to minimize $\eta_1 = |\lambda_1 - r_1/p|$ ($\eta_{1,\min} < 1/p$) and $\eta_2 = |\lambda_2 - r_2/p|$ ($\eta_{2,\min} < 1/p$). Next, consider the toral quantizations $\hat{U}_p(\mathbf{0})$ of the Kronecker maps for $\lambda_1 = \lambda_{1,p} = r_1(p)/p$, $\lambda_2 = \lambda_{2,p} = r_2(p)/p$, and $\boldsymbol{\theta} = \mathbf{0}$. Then $\hat{U}_{\lambda_1,\lambda_2} \equiv \lim_{p\to\infty} \hat{U}_p(\mathbf{0})$ defines a quantization of the Kronecker map with the given irrational values of λ_1 and λ_2 in the classical limit $p \to \infty$. It is easy to show [41] that $\hat{U}_{\lambda_1,\lambda_2}$ is uniquely ergodic quantally, like the quantized ISTs.

8. Conclusions

Generic canonical maps on a two-torus can be quantized only if two conditions are satisfied: (a) Planck's constant assumes rational values, $h = q/p$, where the integer q (the area of the admissible torus $\mathbb{T}^2_Q$ on which the quantization is performed) must satisfy a number-theoretical condition (see theorem 2 in section 3.1). This condition involves only $\mathrm{Tr}(A)$, where A [$\in SL(2, \mathbb{Z})$] defines the linear component of the map. It turns out then that for $\mathrm{Tr}(A) \neq 2$ *not all* integers q coprime to p are allowed. (b) There *exist* allowed quantum boundary conditions (BCs) on $\mathbb{T}^2_Q$. The equation determining these BCs in the most general case involves the linear and translational components of the map. It follows from this equation that allowed BCs always exist if $\mathrm{Tr}(A) \neq 2$, which, curiously, is precisely the case where not all q coprime to p are allowed (see above).

Allowed BCs may not exist if $\mathrm{Tr}(A) = 2$ and the map exhibits a *nonzero* translational component. Representative examples are the irrational skew translations and Kronecker maps. Recent quantization schemes [40, 41] for these maps can be easily understood in the general-BCs framework.

An interesting problem, to be considered in future studies, is how to approach the generic case of irrational values of h for $\mathrm{Tr}(A) \neq 2$ using rational approximants q/p, where q is restricted by the number-theoretical condition in theorem 2.

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Appendix. q-admissibility for integer Q (proof of theorem 2)

We present here the proof of theorem 2 (stated in section 3.1), due to Z Rudnick. This theorem is an immediate consequence of the lemmas below. In what follows, A denotes a matrix in $SL(2, \mathbb{Z})$, Q stands for an integer 2×2 matrix and q is an integer number. We start with some definitions. (a) Given two nonparallel integer vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, a *lattice* Λ is the set of all vectors $\boldsymbol{z_n} = n_1\boldsymbol{x} + n_2\boldsymbol{y} = Q \cdot \boldsymbol{n}$, where $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$ and Q is the (nonsingular) matrix whose columns are $\boldsymbol{x}$ and $\boldsymbol{y}$. The pair $(\boldsymbol{x}, \boldsymbol{y})$ is a *basis* of Λ. (b) Given a matrix A, a lattice Λ is said to be *A-invariant* $(A \cdot \Lambda = \Lambda)$ if, for all $\boldsymbol{n} \in \mathbb{Z}^2$, $A \cdot \boldsymbol{z_n} = \boldsymbol{z_{n'}} \in \Lambda$. Clearly, a lattice Λ defined by the matrix Q is A-invariant $\Longleftrightarrow$ $Q^{-1}AQ$ is integer. (c) A matrix A is *q-admissible* if there exists a matrix Q with $\det(Q) = q$ such that $Q^{-1}AQ$ is integer. In other words, A is q-admissible if there exists an A-invariant sublattice of index q in $\mathbb{Z}^2$. (d) A matrix Q is *primitive* if it is not a proper multiple of an integer matrix. (e) A matrix A is *primitively q-admissible* if there exists a primitive matrix Q with $\det(Q) = q$ such that $Q^{-1}AQ$ is integer.

Lemma 1. *A is q-admissible $\Longleftrightarrow$ A is primitively q'-admissible for some integer q' such that $q = r^2q'$, r integer.*

Proof. If A is primitively q'-admissible, i.e. there exists a primitive integer matrix Q' with $\det(Q') = q'$ such that $Q'^{-1}AQ'$ is integer, then, by replacing Q' by rQ' (r arbitrary nonzero integer), we see that A is q-admissible ($q = r^2q'$). Assume now that A is q-admissible with an A-invariant lattice Λ (a sublattice of index q in $\mathbb{Z}^2$). By the theorem on elementary divisors (see, e.g., chapter 3.7 in [51]), there exists a basis $(\boldsymbol{x}, \boldsymbol{y})$ of $\mathbb{Z}^2$ and positive integers r, s with $r|s$ and $rs = q$, such that $(r\boldsymbol{x}, s\boldsymbol{y})$ is a basis of Λ. Since Λ is A-invariant, we must have $A \cdot (r\boldsymbol{x}) \in \Lambda$ or $A \cdot (r\boldsymbol{x}) = \alpha r\boldsymbol{x} + \beta s\boldsymbol{y}$, for some integers α, β. Dividing the last equation by r, we get $A \cdot \boldsymbol{x} = \alpha\boldsymbol{x} + \beta(s/r)\boldsymbol{y}$. Since s/r is integer ($r|s$), this implies that $(\boldsymbol{x}, s\boldsymbol{y}/r)$ is a basis of an A-invariant sublattice Λ' of index $q' = s/r = q/r^2$ in $\mathbb{Z}^2$. Now, the integer matrix Q' defining Λ' is primitive since one column of Q' is $\boldsymbol{x}$, which, being an element of a basis of $\mathbb{Z}^2$, must be a primitive vector (i.e. an integer vector with coprime components). Thus, A is primitively q'-admissible. □

Lemma 2. *A is primitively q-admissible if and only if A has a primitive eigenvector modulo q, that is a primitive vector $\boldsymbol{x}$ such that $A \cdot \boldsymbol{x} = \alpha\boldsymbol{x}$ mod q, α integer.*

Proof. If A is primitively q-admissible, there exists a basis $(\boldsymbol{x}, \boldsymbol{y})$ of $\mathbb{Z}^2$ such that $(\boldsymbol{x}, q\boldsymbol{y})$ is a basis for an A-invariant lattice Λ (see proof of lemma 1). Then $A \cdot \boldsymbol{x} = \alpha\boldsymbol{x} + \beta q\boldsymbol{y}$ ($\alpha, \beta \in \mathbb{Z}$), or $A \cdot \boldsymbol{x} = \alpha\boldsymbol{x}$ mod q; since $\boldsymbol{x}$ is an element of a basis for $\mathbb{Z}^2$, it must be a primitive vector. Conversely, if A has a primitive eigenvector $\boldsymbol{x}$ modulo q, we can complete $\boldsymbol{x}$ to a basis $(\boldsymbol{x}, \boldsymbol{y})$ of $\mathbb{Z}^2$. The lattice with basis $(\boldsymbol{x}, q\boldsymbol{y})$ is then an A-invariant sublattice of index q in $\mathbb{Z}^2$ and, obviously, is associated with a primitive integer matrix Q. □

From this lemma we immediately deduce:

Lemma 3. *If A is primitively q-admissible, it is primitively d-admissible for any divisor d of q. In particular, d can be the square-free part of q.*

Lemma 4. *Assume that q is a square-free integer. Then A is primitively q-admissible if and only if it is (primitively) p-admissible for every prime divisor p of q.*

Proof. Clearly, $q = \prod_j p_j$, where p_j are distinct primes. Because of lemma 3, we have only to show that if A has primitive eigenvectors $\boldsymbol{x}_j$ modulo p_j, it has a primitive eigenvector $\boldsymbol{x}$ modulo q. Thus, suppose that $\boldsymbol{x}_j$ is a primitive vector such that $A \cdot \boldsymbol{x}_j = \alpha_j \boldsymbol{x}_j \bmod p_j$, $\alpha_j \in \mathbb{Z}$. By the Chinese remainder theorem, there exist $\boldsymbol{x} \in \mathbb{Z}^2$ and $\alpha \in \mathbb{Z}$, unique modulo q, such that $\boldsymbol{x} = \boldsymbol{x}_j \bmod p_j$ and $\alpha_j = \alpha \bmod p_j$. Then $A \cdot \boldsymbol{x} = \alpha \boldsymbol{x} \bmod p_j$, that is $A \cdot \boldsymbol{x} = \alpha \boldsymbol{x} \bmod q$. If $\boldsymbol{x}$ is a primitive vector, the proof is complete. Otherwise, let $\boldsymbol{x} = d\boldsymbol{x}'$, where $d\,(>1)$ is some integer and $\boldsymbol{x}'$ is primitive. We observe that d must be coprime to q, since if some p_j divides both d and q then $\boldsymbol{x}_j = \boldsymbol{0} \bmod p_j$ and so $\boldsymbol{x}_j$ is nonprimitive. Now, $dA \cdot \boldsymbol{x}' = A \cdot (d\boldsymbol{x}') = A \cdot \boldsymbol{x} = \alpha \boldsymbol{x} \bmod q = d\alpha \boldsymbol{x}' \bmod q$. Thus, $dA \cdot \boldsymbol{x}' = d\alpha \boldsymbol{x}' \bmod q$, and since d is coprime to q, we can multiply by its inverse modulo q to recover $A \cdot \boldsymbol{x}' = \alpha \boldsymbol{x}' \bmod q$ with $\boldsymbol{x}'$ primitive. □

Lemma 5. *(i) Let p be an odd prime. Then A is p-admissible $\Longleftrightarrow$ $\mathrm{Tr}(A)^2 - 4$ is a square modulo p. (ii) A is 2-admissible $\Longleftrightarrow$ $\mathrm{Tr}(A)$ is even.*

Proof. We first show that A is p-admissible $\Longleftrightarrow$ $\det(A - \alpha I) = 0 \bmod p$, $\alpha \in \mathbb{Z}$. Assume that A has a primitive eigenvector modulo the prime p, with integer eigenvalue α. It follows then immediately that $\det(A - \alpha I) = 0 \bmod p$. Conversely, assume that $\det(A - \alpha I) = 0 \bmod p$, α integer. Since $\mathbb{Z}/p\mathbb{Z}$ is a field, this implies that there exists an integer vector $\boldsymbol{x} \neq \boldsymbol{0} \bmod p$ such that $A \cdot \boldsymbol{x} = \alpha \boldsymbol{x} \bmod p$. If $\boldsymbol{x}$ is nonprimitive, then $\boldsymbol{x} = d\boldsymbol{x}'$, where $d\,(>1)$ is an integer not divisible by p (otherwise, $\boldsymbol{x} = \boldsymbol{0} \bmod p$) and $\boldsymbol{x}'$ is primitive. Then, by multiplying both sides of $A \cdot \boldsymbol{x} = \alpha \boldsymbol{x} \bmod p$ by the inverse of d modulo p, we get $A \cdot \boldsymbol{x}' = \alpha \boldsymbol{x}' \bmod p$ with $\boldsymbol{x}'$ primitive.

Next, $\det(A - \alpha I) = 0 \bmod p$ is equivalent to $\alpha^2 - k\alpha + 1 = 0 \bmod p$, $k = \mathrm{Tr}(A)$. For $p = 2$, the latter equation has an integer solution α if and only if k is even. For $p > 2$, the equation is equivalent to $4(\alpha^2 - k\alpha + 1) = 0 \bmod p$ (since 4 is invertible modulo p). But $4(\alpha^2 - k\alpha + 1) = (2\alpha - k)^2 - k^2 + 4 = 0 \bmod p$ if and only if $k^2 - 4$ is a square modulo p. □

From lemmas 1 and 3 it follows that A is q-admissible $\Longleftrightarrow$ A is primitively q'-admissible, where q' is the square-free part of q. Lemmas 4 and 5 then complete the proof of theorem 2. □

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