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Vortex structure and characterization of quasiperiodic functions

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Abstract

Quasiperiodic functions (QPFs) are characterized by their full vortex structure in one unit cell. This characterization is much finer and more sensitive than the topological one given by the total vorticity per unit cell (the 'Chern index'). It is shown that QPFs with an arbitrarily prescribed vortex structure exist by constructing explicitly such a 'standard' QPF. Two QPFs with the same vortex structure are *equivalent*, in the sense that their ratio is a function which is strictly periodic, nonvanishing and at least continuous. A general QPF can then be approximately reconstructed from its vortex structure on the basis of the standard QPF and the equivalence concept. As another application of this concept, a simple method is proposed for calculating the quasiperiodic eigenvectors of periodic matrices. Possible applications to the quantum-chaos problem on a phase-space torus are briefly discussed.

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1. Introduction

In this paper we study basic and physically relevant aspects of quasiperiodic functions (QPFs) and derive exact results for them. A QPF is defined here as a complex smooth function $F(x, y)$, periodic up to phase factors in the real variables x and y:

$$F(x+1, y) = \exp[\mathrm{i}\alpha(x, y)]F(x, y) \tag{1}$$

$$F(x, y+1) = \exp[\mathrm{i}\beta(x, y)]F(x, y) \tag{2}$$

where the phases α and β are real smooth functions of x and y and the basic cell of quasiperiodicity is chosen, for simplicity and without loss of generality, as the unit torus. QPFs emerge in a natural way in a variety of physical problems. We mention here, for example, the quasiperiodic solutions of the Ginzburg–Landau equations for the order parameter in superconductivity near the upper critical field [1]; quantum-mechanical representations

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based on a lattice in phase space [2], which turn out to be most natural for describing the dynamics of electrons in solids; magnetic Bloch functions [3–10] and quantum-dynamical eigenstates on a phase-space torus [11–19]; quasiperiodic functions for the fractional quantum Hall effect [20, 21]; and quasiperiodic wave fields introduced recently in Fourier optics [22, 23].

Let us recall here briefly some known properties of QPFs (see more details in section 2 and in the appendix). Due to the single-valuedness of a QPF, the total phase change of $F(x, y)$ by going around the unit-cell boundary in, say, the counterclockwise direction must be an integer multiple N of 2π. This phase change originates from the vortices of $F(x, y)$, which are phase singularities located at the zeros of F, with equiphase lines radiating out from each zero. For a simple (first-order) zero, the phase circulates around the zero by 2π counterclockwise (vortex sign +1) or clockwise (vortex sign -1). A zero of order $n > 1$ may be viewed as the coincidence of n simple vortices, generally of different signs (generically, however, the zeros are simple [24]). Then, if the number of positive (negative) vortices in one unit cell is N_+ (N_-), one must have $N = N_+ - N_-$. The integer N is thus the total vorticity in one unit cell and is an important topological characterization of a QPF.

For example, the total vorticity N of a magnetic Bloch state in the reciprocal-space cell (the 'magnetic Brillouin zone') is the integer quantum Hall conductance (in units of e^2/h) carried by the corresponding magnetic band [3–5, 7, 8]. The analogous quantity for quantum-dynamical eigenstates on a phase-space torus is the Chern index [13–17, 25], which measures the sensitivity of an eigenstate to variations in the toral boundary conditions and reflects some quantum signatures of order and chaos in a semiclassical regime. The coherent-state representation of a toral state [11] is a QPF $F(x, y)$ whose vortices are all *positive* ($N = N_+$) and *isotropic*, i.e. the equiphase lines in the infinitesimal vicinity of a zero are distributed uniformly around it. In this case, N is the dimension of the toral Hilbert space and gives the scaled Planck's constant $h = 1/N$. In a semiclassical regime ($N \gg 1$), the N zeros of a toral eigenstate are distributed along lines for integrable systems and are spread almost uniformly on the torus for chaotic systems [11]. A remarkable fact is that the N zeros *completely determine* $F(x, y)$ (up to a constant normalization factor) [11], so that they provide a good representation (the 'stellar' representation [11, 26]) of toral states (see section 4.2).

In addition to this fact, one should observe that: (a) vortices are associated with several important physical properties and their structure can be measured experimentally [1, 23]; (b) being a topological quantity, the total vorticity is not sensitive to small changes in the system. All these observations motivate one to characterize *general* QPFs by their vortex structure in a more detailed fashion than by just the total vorticity. One is then led to ask to what extent a general QPF can be determined from such a characterization. These are the main issues addressed in this paper. We shall take into account all the relevant characteristics of the vortex structure: the location of each vortex (zero) in the unit cell, its sign, the eccentricity of the ellipse describing the anisotropy of equiphase lines around the vortex and the orientation of the ellipse axes [27] (see section 2). We first show that a QPF with an *arbitrarily prescribed* vortex structure *exists* by constructing it explicitly using Jacobi ϑ_3 functions [28, 29]; we call it the 'standard' QPF. Some of its properties are studied and the case of positive isotropic vortices is investigated in detail. We then show that two QPFs with the same vortex structure are necessarily *equivalent*, i.e. (by definition) their ratio is a function which is strictly periodic, nonvanishing and at least continuous. Thus, a general QPF can be approximately 'reconstructed' from its vortex structure on the basis of the standard QPF and the equivalence concept. As another application of this concept, we propose a simple method for calculating the continuous quasiperiodic eigenvectors of periodic matrices. Such matrices arise in several physical contexts, e.g., Bloch electrons in a magnetic field [3, 4, 7] and quantum dynamics on a phase-space torus [12, 13, 15]. The continuity of the eigenvectors is required, e.g., in the construction of an orthogonal set of localized orbitals in a magnetic field ('magnetic Wannier functions') [6, 7, 10]. Possible applications to the quantum-chaos problem on a phase-space torus are briefly discussed.

This paper is organized as follows. In section 2 and in the appendix, we summarize basic known properties of QPFs and of general vortex structure. In section 3, an exact closed expression for the phase of the ϑ_3 function is derived and some of its properties are studied. In section 4, we explicitly construct a 'standard' QPF having an arbitrarily prescribed vortex structure and study in detail the special case of positive isotropic vortices. In section 5, we introduce and discuss the concept of equivalence of QPFs; the reconstruction of a QPF from its vortex structure is described. In section 6, we propose a simple method for calculating the quasiperiodic eigenvectors of periodic matrices on the basis of the equivalence concept. Conclusions are presented in section 7, which also includes a brief discussion of possible applications to the quantum-chaos problem on a phase-space torus.

2. Quasiperiodic functions and vortex structure

Given a QPF with total vorticity $N = N_+ - N_-$ in one unit cell (see the introduction), it is always possible to make a phase transformation $F(x, y) \to F_N(x, y) = \exp[-\mathrm{i}\varphi(x, y)]F(x, y)$ so that the quasiperiodicity conditions (1) and (2) for $F_N(x, y)$ assume the simple form:

$$F_N(x+1, y) = F_N(x, y) \tag{3}$$

$$F_N(x, y+1) = \exp(-2\pi\mathrm{i}Nx)F_N(x, y). \tag{4}$$

The proof of this is given in the appendix. We shall therefore restrict ourselves from now on to the class of QPFs satisfying (3) and (4). One can easily derive (see [22] and the appendix) a general and useful expression for $F_N(x, y)$ in the case of $N \neq 0$:

$$F_N(x, y) = \sum_{j=0}^{|N|-1} \sum_{n=-\infty}^{\infty} \exp\left[2\pi\mathrm{i}n\left(x + \frac{j}{|N|}\right)\right] \phi_j\left(y + \frac{n}{N}\right) \tag{5}$$

where the $|N|$ functions $\phi_j(y)$, $j = 0, \ldots, |N| - 1$, are uniquely determined by $F_N(x, y)$. For $N = 0$, $F_N(x, y)$ is strictly periodic and can be expanded as a two-dimensional Fourier series. One can express (5) in the equivalent, 'dual' form [22]:

$$F_N(x, y) = \exp(-2\pi\mathrm{i}Nxy) \sum_{j=0}^{|N|-1} \sum_{n=-\infty}^{\infty} \exp\left[2\pi\mathrm{i}n\left(y + \frac{j}{|N|}\right)\right] \tilde{\phi}_j\left(-x + \frac{n}{N}\right) \tag{6}$$

where

$$\tilde{\phi}_j(x) = \sum_{s=0}^{|N|-1} \int_{-\infty}^{\infty} \mathrm{d}y \exp\left[2\pi\mathrm{i}(js/N - Nxy)\right] \phi_s(y). \tag{7}$$

The functions $\{\phi_j(y)\}_{j=0}^{|N|-1}$ or $\{\tilde{\phi}_j(x)\}_{j=0}^{|N|-1}$ are usually localized and may be interpreted as the 'Wannier functions' [10, 22] associated with the 'extended' function $F_N(x, y)$.

We now consider basic properties of the vortex structure of general complex functions $f(x, y)$ (see, e.g., [27]). We express $f(x, y)$ in terms of its real and imaginary parts, $f(x, y) = R(x, y) + \mathrm{i}I(x, y)$, and focus on a vortex located at a simple zero (x_0, y_0) of $f(x, y)$ (an intersection of the curves defined by $R(x, y) = 0$ and $I(x, y) = 0$). For simplicity and without loss of generality, we assume that $(x_0, y_0) = (0, 0)$. Then, sufficiently close to this zero, one has

$$f(x, y) = \rho\exp(\mathrm{i}\chi) \approx R_x x + R_y y + \mathrm{i}(I_x x + I_y y) \tag{8}$$

where (ρ, χ) are the modulo and phase of $f(x, y)$, and $R_x = \partial R/\partial x|_{(x,y)=(0,0)}$, etc. The contours of constant ρ are approximately ellipses,

$$\rho^2 \approx ax^2 + 2cxy + by^2 \tag{9}$$

where $a = R_x^2 + I_x^2$, $b = R_y^2 + I_y^2$ and $c = R_x R_y + I_x I_y$. The shape and orientation of the ellipse (9) are fully determined by the eigenvalues and eigenvectors of the symmetric matrix $(a, c; c, b)$. The eigenvalues $\lambda_{1,2}$ $(\lambda_1 \leqslant \lambda_2)$ give the eccentricity ε of the ellipse while the orthogonal eigenvectors are in the direction of the ellipse axes:

$$\varepsilon = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} = \frac{1}{\sqrt{2}|\omega|}(d^2 - 4\omega^2)^{1/4}[d - (d^2 - 4\omega^2)^{1/2}]^{1/2} \tag{10}$$

$$\tan(2\theta) = \frac{2c}{a - b} \tag{11}$$

where $\omega = R_x I_y - R_y I_x$, $d = a + b$ and θ is the angle between an axis of the ellipse and the x axis. Concerning the phase χ of $f(x, y)$, a straightforward calculation using (8), (10) and (11) yields

$$\frac{d\chi}{d\gamma} \approx \frac{\pi\eta}{A} r^2 = \frac{\eta}{(1-\varepsilon^2)^{1/2}\cos^2(\gamma-\theta) + (1-\varepsilon^2)^{-1/2}\sin^2(\gamma-\theta)} \tag{12}$$

where (r, γ) are the modulo and phase of $z = x + \mathrm{i}y$, $\eta = \mathrm{sgn}(\omega)$ and $A = \pi\rho^2/|\omega|$ is the ellipse area. In equation (12), $r = r(\gamma)$ gives the 'polar' plot of the ellipse (9). It is then easy to check that $\int_0^{2\pi} d\gamma(d\chi/d\gamma) = 2\pi\eta$, so that $\eta = \mathrm{sgn}(\omega)$ is just the sign of the vortex. The main and important conclusion from (12) is: *the distribution of equiphase lines in the infinitesimal vicinity of a vortex is completely determined by its sign (η) and by the shape (ε) and orientation (θ) of the associated ellipse*. In particular, this distribution is uniform (constant $d\chi/d\gamma$) if and only if the ellipse is a circle ($\varepsilon = 0$, an *isotropic* vortex).

3. Phase of ϑ_3 function

The Jacobi ϑ_3 function [28, 29],

$$\vartheta_3(z|\tau) \equiv \sum_{n=-\infty}^{\infty} \exp[\mathrm{i}(\pi\tau n^2 + 2nz)] \qquad \mathrm{Im}\,\tau > 0 \tag{13}$$

is an entire analytic function of order 2 in the complex variable z and is characterized by the parameter τ. It satisfies the quasiperiodicity conditions [28]:

$$\vartheta_3(z + \pi|\tau) = \vartheta_3(z|\tau) \qquad \vartheta_3(z + \pi\tau|\tau) = \exp[-\mathrm{i}(2z + \pi\tau)]\vartheta_3(z|\tau). \tag{14}$$

Function (13) emerges naturally in many physical problems exhibiting some kind of two-dimensional periodicity [1, 11, 13, 20–23] and has a quite simple vortex structure (see below). It is instructive to understand how this structure affects the phase of (13). We derive here an exact closed expression for this phase and study some of its properties.

We start from the formula

$$\ln\frac{\vartheta_3(z|\tau)}{\vartheta_3(0|\tau)} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{q^n}{1 - q^{2n}}\sin^2(nz) \qquad q \equiv \exp(\mathrm{i}\pi\tau) \tag{15}$$

which is a special case of formula 16.30.3 in [29]. It is easy to see that the expansion (15) converges only for

$$|\,\mathrm{Im}\, z| \leqslant \frac{\pi}{2}\,\mathrm{Im}\,\tau. \tag{16}$$

To extend equation (15) beyond the strip (16), we first express z as $z = \tilde{z} + m\pi\tau$, where $-(\pi/2)\,\mathrm{Im}\,\tau \leqslant \mathrm{Im}\,\tilde{z} < (\pi/2)\,\mathrm{Im}\,\tau$ and m is some integer (the pair $(\tilde{z}, m)$ is uniquely determined by z). Using then the quasiperiodicity conditions (14), we easily find that

$$\ln\frac{\vartheta_3(\tilde{z}+m\pi\tau|\tau)}{\vartheta_3(0|\tau)} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^n}{1-q^{2n}}\sin^2(n\tilde{z}) - \mathrm{i}(2m\tilde{z}+m^2\pi\tau). \qquad (17)$$

We now express $\tilde{z}$ and τ in terms of their real and imaginary parts as follows: $\tilde{z} = \pi(\tilde{x}+\mathrm{i}\tilde{y})$ and $\tau = \tau_x + \mathrm{i}\tau_y$. Taking the imaginary part of (17), we then obtain the following general expression for the phase of the ϑ_3 function:

$$\begin{aligned}\arg\vartheta_3(\tilde{z}+m\pi\tau|\tau) = {}& 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{q^n}{1-q^{2n}}\sin(2\pi n\tilde{x})\sinh(2\pi n\tilde{y}) \\ & - 2\pi m\tilde{x} - m^2\pi\tau_x + \arg\vartheta_3(0|\tau).\end{aligned} \qquad (18)$$

The constant phase $\arg\vartheta_3(0|\tau)$ has to be calculated separately, but it vanishes in some cases (e.g., for imaginary τ, see below).

The phase (18) should be singular at a vortex (zero) of $\vartheta_3(z|\tau)$. It is well known [28] that the zeros of $\vartheta_3(z|\tau)$ are simple and form a lattice,

$$z = z_{l,m} = \left(l-\tfrac{1}{2}\right)\pi + \left(m-\tfrac{1}{2}\right)\pi\tau \qquad (19)$$

for all integer pairs (l, m), so that one has precisely one zero per unit cell of quasiperiodicity (see (14)). The singularity of (18) at any of the points (19) can be clearly exhibited in the case of imaginary τ ($\tau = \mathrm{i}\tau_y$). Consider in this case the 'horizontal' lines $z = \pi(x - \tau/2 + m\tau)$ ($x = \tilde{x}$), corresponding to $\tilde{z} = -\pi\tau/2$. We easily find from (18) that on these lines

$$\arg\vartheta_3(\pi x - \pi\tau/2 + m\pi\tau|\tau) = \pi\,\mathrm{saw}(x) - 2\pi m x \qquad (20)$$

where the sawtooth function $\mathrm{saw}(x) \equiv x$ for $-1/2 \leqslant x < 1/2$ and is periodically continued beyond this interval (note that $\arg\vartheta_3(0|\tau) = 0$ in this case since $\vartheta_3(0|\tau)$ is real and positive for imaginary τ). Clearly, (20) is discontinuous at a zero (19) as z is varied in the horizontal (x) or in the vertical (y) direction. The magnitude of the discontinuity in both directions is π. This discontinuity is due to the passage from one equiphase line to the opposite one through an *isotropic* vortex (19). We recall here that a zero z_0 of an analytic function $\Phi(z)$ is an isotropic and positive vortex because sufficiently close to z_0 one has $\Phi(z) \approx \Phi'(z_0)(z-z_0) = |\Phi'(z_0)(z-z_0)|\exp[\mathrm{i}(\gamma+\gamma_0)]$, where γ, γ_0 and $\gamma+\gamma_0 \approx \chi$ are the phases of $z-z_0$, $\Phi'(z_0)$ and $\Phi(z)$, respectively. Figure 1 shows the phase plot of $\vartheta_3[\pi(x+\mathrm{i}y)|\mathrm{i}]$ in two unit cells.

4. Standard quasiperiodic function

4.1. General

We construct here explicitly a 'standard' QPF $F_{\mathrm{S},\mathcal{V}}(x, y)$ having an arbitrarily prescribed vortex structure $\mathcal{V}$: $N_\mathrm{v} = N_+ + N_-$ vortices in one unit cell, with locations $\{(x_{0,i}, y_{0,i})\}_{i=1}^{N_\mathrm{v}}$ and characteristics $\{\eta_i, \varepsilon_i, \theta_i\}_{i=1}^{N_\mathrm{v}}$ (see section 2). Some properties of the standard QPF are studied and the case of positive and isotropic vortices ($\eta_i = 1$, $\varepsilon_i = 0$ for all i) is considered in detail in section 4.2.

We start by considering the function

$$F_\eta(x, y) = \sum_{n=-\infty}^{\infty}\exp(2\pi\mathrm{i}nx)\phi(y+\eta n) \qquad (21)$$

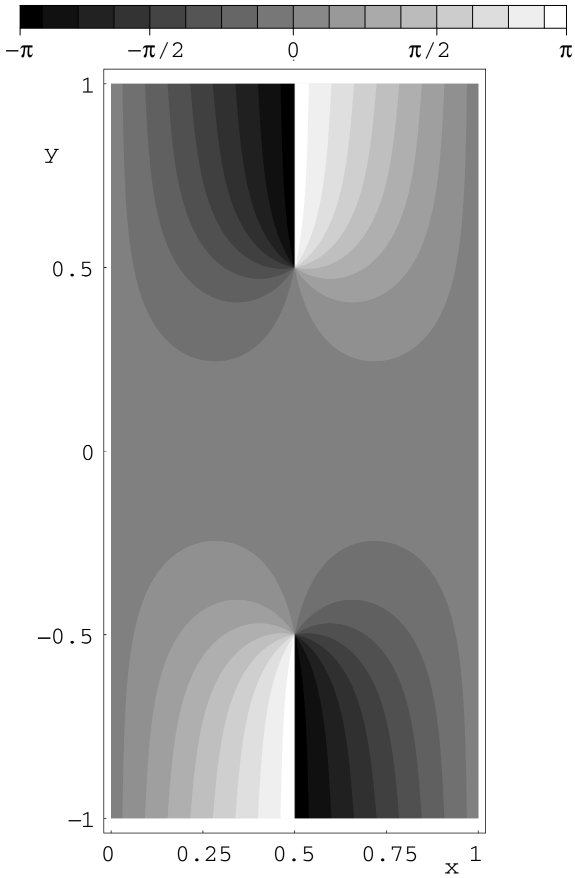

Figure 1. Phase of $\vartheta_3[\pi(x+\mathrm{i}y)|\mathrm{i}]$ in two unit cells of quasiperiodicity, with one vortex per unit cell. The vortices, located at $(x, y) = (0.5, \pm 0.5)$, are positive and isotropic.

where $\eta = \pm 1$,

$$\phi(y) = \exp(-\pi g y^2 - 2\pi \mathrm{i} w y) \tag{22}$$

and g and w are complex numbers with Re $g > 0$. Function (21) is a QPF with total vorticity $N = \eta = \pm 1$ in one unit cell (compare with (5)). We show that (21) has precisely *one* zero with vorticity η in the unit cell, and we shall find its location (x_0, y_0) and characteristics (ε, θ) in terms of g and w. Inserting (22) in (21), we immediately get

$$F_\eta(x, y) = \phi(y)\vartheta_3[\pi(x + \mathrm{i}\eta g y - \eta w)|\mathrm{i}g] \tag{23}$$

where $\vartheta_3(z|\tau)$ is the ϑ_3 function (13). Let us express g and w in terms of their real and imaginary parts: $g = g_1 + \mathrm{i}g_2$ $(g_1 > 0)$ and $w = w_1 + \mathrm{i}w_2$. Using the formula (19) for the zeros of $\vartheta_3(z|\tau)$, we easily find that the zeros of (23) are (x_l, y_m), where

$$x_l = l - \frac{1}{2} + \eta\left(w_1 + \frac{w_2 g_2}{g_1}\right) \qquad y_m = \eta\left(m - \frac{1}{2}\right) + \frac{w_2}{g_1} \tag{24}$$

for all integer pairs (l, m). It is clear from (24) that there is precisely one zero (x_0, y_0) in one unit cell, so that its vorticity must be $N = \eta$. It is also clear that this zero can be positioned arbitrarily in the unit cell just by varying w.

Next, let us expand (23) around a zero. Using the notation $\delta x = x - x_l$, $\delta y = y - y_m$ and $D = \pi\phi(y_m)\vartheta_3'[\pi(x_l + \mathrm{i}\eta g y_m - \eta w)|\mathrm{i}g]$, where $\vartheta_3'(z|\tau)$ is the derivative of the ϑ_3 function, we obtain, to first order in δx and δy,

$$F_\eta(x, y) = \rho\exp(\mathrm{i}\chi) \approx D(\delta x - \eta g_2\delta y + \mathrm{i}\eta g_1\delta y)$$

so that the contours of constant ρ around a zero are approximately given by

$$\frac{\rho^2}{|D|^2} \approx (\delta x)^2 - 2\eta g_2\delta x\delta y + \left(g_1^2 + g_2^2\right)(\delta y)^2. \tag{25}$$

By comparing (25) with (9) and using relations (10) and (11), we can write the expressions for (ε, θ) in terms of η and g. These expressions can be inverted by a straightforward but lengthy calculation. The final result is

$$g_1 = \frac{(1-\varepsilon^2)^{1/2}}{1-\varepsilon^2\cos^2(\theta)} \qquad g_2 = -\frac{\eta\varepsilon^2\sin(\theta)\cos(\theta)}{1-\varepsilon^2\cos^2(\theta)}. \tag{26}$$

One should note that for a given ellipse (25), θ assumes two values giving the orientations of the two axes and differing by $\pi/2$ (see relation (11)). The meaning of the corresponding values of g_1 in (26) can be easily understood by considering the simple case of $\theta = 0, \pi/2$. In this case, $g_2 = 0$ and (22) represents a coherent state with width $(g_1)^{-1/2}$. It is easy to check from (26) that $g_1(\theta = \pi/2) = 1/g_1(\theta = 0)$. Using this in the dual expression (6) with (7), it follows that if $\theta = 0$ is associated with (21), $\theta = \pi/2$ is associated with the $\pi/2$-rotated QPF $F_\eta(y, -x)$. Then, the relation $g_1(0)g_1(\pi/2) = 1$ expresses just the minimal product of quantum uncertainties for coherent states in the (x, y) 'phase space'. The corresponding 'squeezing parameter' [30] is $g_1(\pi/2)/g_1(0) = 1 - \varepsilon^2$.

We thus see that it is possible to construct an 'elementary' QPF (21) with precisely one simple zero in the unit cell, with arbitrary vortex characteristics. The latter determine uniquely (21) with (22). The 'standard' QPF will then be defined as the product of N_v elementary QPFs:

$$F_{\mathrm{S},\mathcal{V}}(x, y) = \prod_{i=1}^{N_\mathrm{v}} F_{\eta_i}(x, y) \tag{27}$$

where $F_{\eta_i}(x, y)$, $i = 1, \ldots, N_\mathrm{v}$, are functions (21) with (22) defined by complex numbers $(g^{(i)}, w^{(i)})$ which are uniquely determined by an arbitrarily prescribed vortex structure $\mathcal{V} = \{(x_{0,i}, y_{0,i}); \eta_i, \varepsilon_i, \theta_i\}_{i=1}^{N_\mathrm{v}}$. Clearly, the function (27) satisfies the quasiperiodicity conditions (3) and (4) with

$$N = \sum_{i=1}^{N_\mathrm{v}} \eta_i.$$

An exact expression for the phase of (27) in terms of the vortex characteristics can be easily written using (23) and expression (18) for the phase of the ϑ_3 function. Figure 2 shows the phase plot of a standard QPF. Figure 3 is a contour plot of the modulo of this QPF, showing how the vortex structure affects the localization properties of (27) in one unit cell.

Let us denote the positive and negative vortices by $(i, +)$ and $(i', -)$, respectively, where $i = 1, \ldots, N_+$ and $i' = 1, \ldots, N_-$, and assume, for definiteness, that $N_+ \geqslant N_-$. Expression (27) can then be written as

$$F_{\mathrm{S},\mathcal{V}}(x, y) = \prod_{i=1}^{N_-} F_{i,+}(x, y)F_{i,-}(x, y) \prod_{i=N_-+1}^{N_+} F_{i,+}(x, y) = F_{\mathrm{S},\mathcal{V}_0}(x, y)F_{\mathrm{S},\mathcal{V}_+}(x, y) \tag{28}$$

where the functions $F_{\mathrm{S},\mathcal{V}_0}(x, y)$ and $F_{\mathrm{S},\mathcal{V}_+}(x, y)$ are defined by the first and second product in (28), respectively. The function $F_{\mathrm{S},\mathcal{V}_0}(x, y)$ is strictly periodic, since every pair $(i, +; i, -)$, $i = 1, \ldots, N_-$, contributes zero vorticity. Thus, the total vorticity $N = N_+ - N_-$ may be

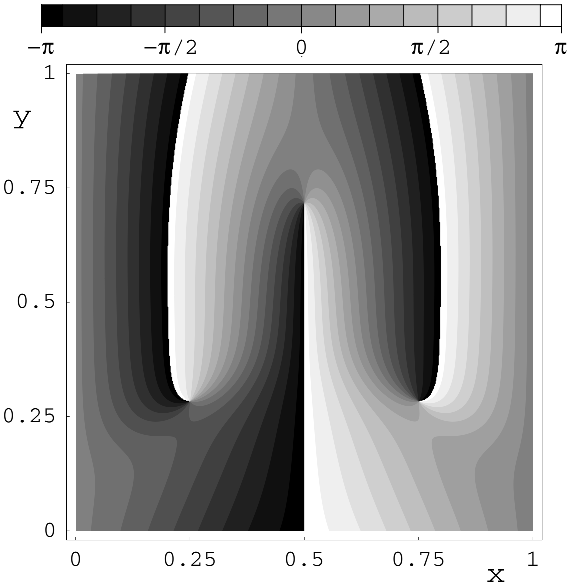

Figure 2. Phase of a standard QPF (27) with three vortices per unit cell. The vortices are located at the corners of an equilateral triangle (at the centre of the unit cell) with side length equal to 0.5. The two vortices at the basis of the triangle are positive while the third vortex is negative, giving a total vorticity $N = 1$. All the vortices are anisotropic with eccentricity $\varepsilon = \sqrt{3}/2$ and the angles θ between the major axis of the associated ellipse and the positive x axis are 30°, 150° and 270°.

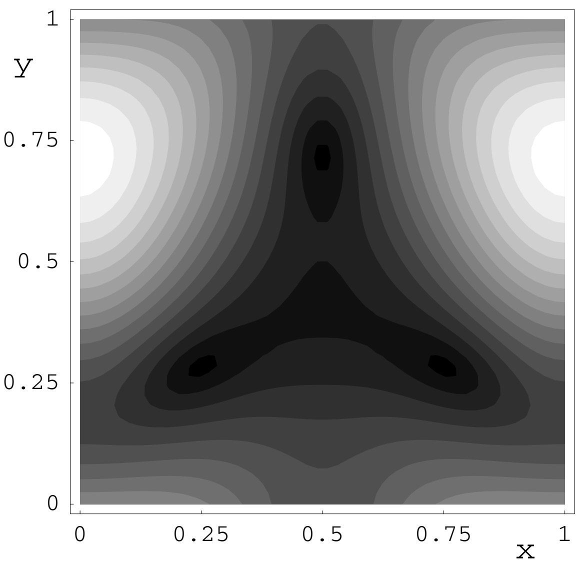

Figure 3. Modulo of the standard QPF whose phase plot is shown in figure 2. Darker regions correspond to lower values of the modulo. Because of the particular orientations of the major axes of the anisotropy ellipses, the modulo assumes its smallest values inside the triangle.

attributed entirely to $F_{\mathrm{S},\mathcal{V}_+}(x, y)$, featuring only the N 'extra' positive vortices. Since the latter vortices can be chosen in different ways, there is an obvious arbitrariness in the factorization (28).

By expanding the products in (28) with (21), the QPFs $F_{S,\mathcal{V}}(x, y)$ and $F_{S,\mathcal{V}_+}(x, y)$ can be expressed both as sums (5) featuring $|N|$ independent Wannier functions $\{\phi_{S,j}(y)\}_{j=0}^{|N|-1}$ and $\{\phi^+_{S,j}(y)\}_{j=0}^{|N|-1}$, respectively. While $|N|$ is equal to the number of elementary QPFs in $F_{S,\mathcal{V}_+}(x, y)$, it is generally much smaller than the corresponding number $N_v = N_+ + N_-$ in a standard QPF $F_{S,\mathcal{V}}(x, y)$. Exact expressions for $\{\phi_{S,j}(y)\}_{j=0}^{|N|-1}$ $\left(\text{or } \{\phi^+_{S,j}(y)\}_{j=0}^{|N|-1}\right)$ can be found easily using standard methods (see, e.g., appendix A in [22]). One can then verify that the localization of the functions $\{\phi_{S,j}(y)\}_{j=0}^{|N|-1}$ is generally similar to that of (22), i.e., they decay faster than exponentially (see the explicit expression (30) below in the case of positive isotropic vortices). The same decay rate is exhibited by the Fourier coefficients of the strictly periodic function $F_{S,\mathcal{V}_0}(x, y)$ $(N = 0)$ which connects $F_{S,\mathcal{V}_+}(x, y)$ with $F_{S,\mathcal{V}}(x, y)$ (see further discussion in section 5).

4.2. Coherent-state representation of toral quantum states

In the case that all the vortices are positive and isotropic ($\eta_i = 1, \varepsilon_i = 0, i = 1, \ldots, N$) with *arbitrary* locations in the unit cell, the standard QPF $F_{S,\mathcal{V}}(x, y)$ is essentially the coherent-state representation of a general quantum state on a phase-space torus [11]. Consider a unit torus $\mathbb{T}^2$ in the (x, y) phase space, where x is position and, for convenience, $-y$ is interpreted as momentum. Quantization on $\mathbb{T}^2$ is possible only if the scaled Planck's constant satisfies $h = 1/p$, where p (an integer) is the number of independent quantum states for given toral boundary conditions; these conditions are specified by a Bloch quasivector $\mathbf{w} = (w_1, w_2)$ ranging in some 'Brillouin zone' [15, 16, 19]. The position (x) representation of a general state $|\Psi_{\mathbf{w}}\rangle$ on $\mathbb{T}^2$ is [18, 19]

$$\langle x|\Psi_{\mathbf{w}}\rangle = \sum_{m=0}^{p-1} \nu(m; \mathbf{w}) \sum_{n=-\infty}^{\infty} \exp(-2\pi \mathrm{i} n w_2)\delta(x - w_1 - m/p - n) \tag{29}$$

where $\{\nu(m; \mathbf{w})\}_{m=0}^{p-1}$ are arbitrary complex coefficients. The coherent-state representation of $|\Psi_{\mathbf{w}}\rangle$ is $\langle x, y|\Psi_{\mathbf{w}}\rangle$, where $\langle x|x', y'\rangle = (\pi\hbar)^{-1/4} \exp\{-[(x - x')^2 + \mathrm{i}y'(2x - x')]/(2\hbar)\}$. After a straightforward calculation using (29), (6) and (7), we find, omitting an uninteresting factor, that $\langle x, y|\Psi_{\mathbf{w}}\rangle = F_N(x, y)$. Here $N = p$ and the QPF $F_N(x, y)$ is given by (5) with

$$\phi_j(y) = \bar{\nu}(j; \mathbf{w}) \exp(-\pi N y^2 - 2\pi \mathrm{i} N w y) \qquad j = 0, \ldots, N - 1 \tag{30}$$

where $\bar{\nu}(j; \mathbf{w}) = \exp[2\pi \mathrm{i}(N - j)w_2]\nu(N - j; \mathbf{w})$ $(\bar{\nu}(0; \mathbf{w}) = \nu(0; \mathbf{w}))$ and $w = w_1 + \mathrm{i}w_2$. More explicitly, we have

$$F_N(x, y) = \exp(-\pi N y^2 - 2\pi \mathrm{i} N w y) \times \sum_{s=0}^{N-1} \xi_s(\mathbf{w}) \exp\left[-\frac{\pi s^2}{N} - 2\pi \mathrm{i}s(z - w)\right] \vartheta_3[\pi N(z - w - \mathrm{i}s/N)|\mathrm{i}N] \tag{31}$$

where $z = x + \mathrm{i}y$ and $\xi_s(\mathbf{w}) = \sum_{j=0}^{N-1} \bar{\nu}(j; \mathbf{w}) \exp(-2\pi \mathrm{i}sj/N)$. Similar to the ϑ_3 function, the sum in (31) is an entire analytic function. Therefore, the QPF (31) must have precisely N positive and isotropic vortices in one unit cell, whose locations we denote by $\{(x_{0,i}, y_{0,i})\}_{i=1}^{N}$. An important result in [11] is that the sum in (31) can be expressed as a product,

$$F_N(x, y) = \exp(-\pi N y^2 - 2\pi \mathrm{i} N w y) \prod_{i=1}^{N} \vartheta_3[\pi(z - w^{(i)})|\mathrm{i}] \tag{32}$$

where $w^{(i)} = x_{0,i} + \mathrm{i}y_{0,i} + (1 + \mathrm{i})/2 \bmod(1, \mathrm{i})$ and one has the condition

$$\frac{1}{N}\sum_{i=1}^{N} w^{(i)} = w. \tag{33}$$

Using relations (22)–(24), it is easy to see that (32) with (33) is just the standard QPF (27) in the case of positive isotropic vortices ($\eta_i = 1$, $\varepsilon_i = 0$ and $g^{(i)} = 1$ for all i). Given N such vortices whose locations are arbitrary apart from condition (33), there always exists a nonzero vector $\{\xi_s(\mathbf{w})\}_{s=0}^{N-1}$ in (31) satisfying the N equations $\{F_N(x_{0,i}, y_{0,i}) = 0\}_{i=1}^{N}$. This is because (33) is precisely the condition for the vanishing of the determinant associated with these equations, as one can verify easily using (14) in (31). We thus see that a toral quantum state is completely determined by its N zeros in the coherent-state representation (equation (32)) and that the locations of these zeros can be specified arbitrarily, corresponding to the arbitrary specification of the N complex coefficients $\{\xi_s(\mathbf{w})\}_{s=0}^{N-1}$. Only $N-1$ of these coefficients are actually independent (since a quantum state is defined up to a constant factor); similarly, there are only $N-1$ independent zeros because of the restriction (33). The zeros thus provide a good representation (the 'stellar' representation [11, 26]) of toral states.

5. Equivalence of quasiperiodic functions

Let us assume that two QPFs, $F_N(x, y)$ and $F'_N(x, y)$, have precisely the same vortex structure $\mathcal{V} = \{(x_{0,i}, y_{0,i}); \eta_i, \varepsilon_i, \theta_i\}_{i=1}^{N_v}$. We show that such QPFs are *equivalent*, i.e. by definition,

$$F_N(x, y) = \psi(x, y) F'_N(x, y) \tag{34}$$

where $\psi(x, y)$ is a strictly periodic, nonvanishing and at least continuous function. The inverse statement is obviously true: The equivalence relation (34) implies that $F_N(x, y)$ and $F'_N(x, y)$ have the same vortex structure. This equivalence concept will then be discussed and applied in this and the next section.

Consider a (common) vortex of $F_N(x, y)$ and $F'_N(x, y)$. For simplicity and without loss of generality, we assume that this vortex is a simple zero located at $(x_0, y_0) = (0, 0)$. We denote the modulo and phase of $F_N(x, y)$ and $F'_N(x, y)$ by (ρ, χ) and (ρ', χ'), respectively. Sufficiently close to the vortex, these quantities satisfy the approximate relations (9) and (12) which become exact in the limit $r \to 0$. It is clear from (10) and (11) that (ε, θ) determine the symmetric matrix $(a, c; c, b)$ in (9) up to a constant positive factor. Since $F_N(x, y)$ and $F'_N(x, y)$ have the same vortex characteristics, it follows that the ratio ρ/ρ' assumes a well-defined positive value ς in the limit $r \to 0$, i.e. ς is independent of the phase γ. Similarly, relation (12) implies that $\lim_{r\to 0} \mathrm{d}(\chi - \chi')/\mathrm{d}\gamma = 0$ or $\lim_{r\to 0}(\chi - \chi') = \chi_0$, a constant phase independent of γ. We thus see that the ratio $F_N(x, y)/F'_N(x, y) = \psi(x, y)$ is well defined at a vortex,

$$\lim_{r\to 0} \psi(x, y) = \varsigma \exp(\mathrm{i}\chi_0) \neq 0 \tag{35}$$

so that $\psi(x, y)$ is a nonvanishing and at least continuous function everywhere. Since both $F_N(x, y)$ and $F'_N(x, y)$ satisfy the quasiperiodicity conditions (3) and (4), $\psi(x, y)$ is also a strictly periodic function.

By combining (34) with the results of the previous section, we see that all the equivalent QPFs with a given, arbitrarily prescribed vortex structure $\mathcal{V}$ form precisely the *equivalence class* $\mathcal{C}_{\mathcal{V}}$ of the standard QPF $F_{\mathrm{S},\mathcal{V}}(x, y)$,

$$\mathcal{C}_{\mathcal{V}} = \{F_N(x, y) = \psi(x, y) F_{\mathrm{S},\mathcal{V}}(x, y)\} \tag{36}$$

for all strictly periodic, nonvanishing and at least continuous functions $\psi(x, y)$. By expanding $\psi(x, y)$ in a Fourier series,

$$\psi(x, y) = \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \psi_{n,n'} \exp[2\pi\mathrm{i}(nx + n'y)] \tag{37}$$

the equivalence relation in (36) can be expressed in the representation of the corresponding Wannier functions $\{\phi_j(y)\}_{j=0}^{|N|-1}$ in (5). After a straightforward but lengthy calculation using (37), we obtain

$$\phi_j(y) = \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \psi_{n,n'} \exp\left\{2\pi \mathrm{i} \left[\frac{jn}{|N|} + n'\left(y + \frac{n}{N}\right)\right]\right\} \phi_{\mathrm{S},(j+\eta n') \bmod |N|}\left(y + \frac{n}{N}\right) \tag{38}$$

$(j = 0, \ldots, |N| - 1)$, where $\eta = \mathrm{sgn}(N)$ and $\{\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$ are the Wannier functions for $F_{\mathrm{S},\mathcal{V}}(x, y)$. As pointed out in section 4, the latter functions can be calculated analytically and their decay rate is faster than exponential. In addition, one should note the following properties of the Fourier coefficients $\psi_{n,n'}$. First, $\psi(x, y) \neq 0$ implies that $\psi_{0,0} \neq 0$. Second, since $\psi(x, y)$ is at least continuous, $\psi_{n,n'}$ exhibit at least a power-law decay, e.g., such as $(nn')^{-2}$, and faster decay rates are easily obtained if $\psi(x, y)$ has continuous derivatives at the vortices. Thus, $\{\phi_j(y)\}_{j=0}^{|N|-1}$ can be generally well approximated in terms of $\{\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$ using only coefficients $\psi_{n,n'}$ with $|n|$ not much larger than $|N|$. If $\psi(x, y)$ is very close to a constant, the approximation $\{\phi_j(y) \approx \psi_{0,0}\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$ may be sufficient.

It is instructive to compare the equivalence relation in (36) with relation (28). Since $F_{\mathrm{S},\mathcal{V}_0}(x, y)$ is strictly periodic like $\psi(x, y)$, the Wannier functions $\{\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$ for $F_{\mathrm{S},\mathcal{V}}(x, y)$ can be expressed in terms of the corresponding functions $\left\{\phi^+_{\mathrm{S},j}(y)\right\}_{j=0}^{|N|-1}$ for $F_{\mathrm{S},\mathcal{V}_+}(x, y)$ by a relation analogous to (38). Now, however, this relation is not invertible since, unlike $\psi(x, y)$, $F_{\mathrm{S},\mathcal{V}_0}(x, y)$ generally vanishes at a finite number $2N_-$ of vortices, N_- positive and N_- negative. As a consequence, $\left\{\phi^+_{\mathrm{S},j}(y)\right\}_{j=0}^{|N|-1}$ *cannot* be expressed in terms of $\{\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$. This reflects the fact that while $F_{\mathrm{S},\mathcal{V}}(x, y)$ and $F_{\mathrm{S},\mathcal{V}_+}(x, y)$ are associated with the same value of the total vorticity N, they are not equivalent since the vortex structures $\mathcal{V}$ and $\mathcal{V}_+$ are different.

As an application of the equivalence concept, let us assume that the vortex structure of a general QPF $F_N(x, y)$, as well as its values on a sufficiently large grid $\mathcal{G}$ of points $(x = l, y = l')/(2L + 1), l, l' = 0, \ldots, 2L$, are known from experimental or numerical measurements. A proper 'reconstruction' of $F_N(x, y)$ from this data can be performed as follows. First, one constructs the standard QPF (27) having the given vortex characteristics of $F_N(x, y)$. Then, the set of values of $F_N(x, y)/F_{\mathrm{S},\mathcal{V}}(x, y)$ on $\mathcal{G}$ is used for a calculation of the leading Fourier coefficients $\psi_{n,n'}$ in (37), $n, n' = -L, \ldots, L$. The resulting truncated Fourier expansion defines a smooth approximation $\tilde{\psi}(x, y)$ of the 'envelope' function $\psi(x, y)$. From the properties above of $\psi_{n,n'}$, we see that the accuracy of this approximation is at least $O(L^{-2})$. Finally, $F_N(x, y)$ is approximated by a QPF $\tilde{F}_N(x, y)$ in the same equivalence class (36), $F_N(x, y) \approx \tilde{F}_N(x, y) = \tilde{\psi}(x, y)F_{\mathrm{S},\mathcal{V}}(x, y)$. Alternatively, one can use relation (38) to obtain good approximations of the Wannier functions $\{\phi_j(y)\}_{j=0}^{|N|-1}$ in terms of the corresponding functions $\{\phi_{\mathrm{S},j}(y)\}_{j=0}^{|N|-1}$ which can be calculated analytically. By construction, $\tilde{F}_N(x, y)$ reproduces properly the essential singular features of $F_N(x, y)$, in particular the phase pattern, sufficiently close to the vortices.

6. Quasiperiodic eigenvectors of periodic matrices

Consider a $p \times p$ Hermitian or unitary matrix M whose elements $\{M_{m,m'}(x, y)\}_{m,m'=0}^{p-1}$ are smooth periodic functions of (x, y) with unit cell $[0, 1)^2$. We assume that the p eigenvalues $\lambda_l(x, y), l = 1, \ldots, p$, are nondegenerate for all (x, y). The 'band' functions $\{\lambda_l(x, y)\}_{l=1}^{p}$ are obviously periodic with unit cell $[0, 1)^2$. However, the normalized eigenvectors $\mathbf{V}_l(x, y) = \{v_l(m; x, y)\}_{m=0}^{p-1}$ (column vectors) are determined up to an arbitrary phase factor, so that they may be only quasiperiodic on $[0, 1)^2$. More precisely, they will satisfy conditions (1) and (2) with phases $\alpha(x, y)$ and $\beta(x, y)$ depending on l but not, of course, on the component m. In particular, all the components of an eigenvector will have the same total vorticity N_l in $[0, 1)^2$. The integer N_l is the *Chern index* associated with band l.

For example, in the context of quantum dynamics on a phase-space torus, the vector $\{v_l(m; \mathbf{w})\}_{m=0}^{p-1}$ ($(x, y) \to \mathbf{w}$) determines a toral eigenstate (29) and is an eigenvector of a $p \times p$ unitary matrix $M(\mathbf{w})$ representing the evolution operator [13, 15]. This matrix is periodic in a Brillouin zone and $\{v_l(m; \mathbf{w})\}_{m=0}^{p-1}$ is generally quasiperiodic in this zone with Chern index N_l reflecting quantum signatures of order and chaos in a semiclassical regime [13–16]. Another example is magnetic Bloch states which are determined by eigenvectors of a Hermitian matrix representing the Hamiltonian [7]. This matrix is periodic in a 'magnetic Brillouin zone' and the Chern index of an eigenvector gives the integer Hall conductance (in units of e^2/h) carried by the corresponding magnetic band [3–5, 7, 8].

It is not clear, *a priori*, how quasiperiodicity can emerge from the diagonalization of a strictly periodic matrix. We give here a simple explanation of the origin of continuous quasiperiodic eigenvectors using the equivalence concept introduced in the previous section. At the same time, a natural method for calculating these eigenvectors will become evident.

To obtain a particular eigenvector $\mathbf{V}(x, y)$ (for simplicity, the band label l is suppressed here), we proceed as follows. First, the band function $\lambda(x, y)$ is calculated as usual by diagonalizing M. Then, the value of one component, say $m = 0$, is fixed to 1. The other components, denoted by $\{\tilde{v}(m; x, y)\}_{m=1}^{p-1}$, satisfy the system of equations $\left\{\sum_{m'=1}^{p-1} \widetilde{M}_{m,m'}(x, y)\tilde{v}(m'; x, y) = -M_{m,0}(x, y)\right\}_{m=1}^{p-1}$, where $\widetilde{M}$ is the $(p-1) \times (p-1)$ matrix $\{M_{m,m'}(x, y) - \lambda(x, y)\delta_{m,m'}\}_{m,m'=1}^{p-1}$. Since $\lambda(x, y)$ is a nondegenerate eigenvalue of M, this system of equations has a unique solution $\{\tilde{v}(m; x, y)\}_{m=1}^{p-1}$ which can be *formally* written as

$$\tilde{v}(m; x, y) = \frac{\Lambda_m(x, y)}{\Lambda_0(x, y)} \tag{39}$$

where $\Lambda_0(x, y)$ is the determinant of $\widetilde{M}$ and $\{\Lambda_m(x, y)\}_{m=0}^{p-1}$ are all strictly periodic and at least continuous functions of (x, y). Using the equivalence relation in (36), we can then write

$$\Lambda_m(x, y) = \psi_m(x, y) F_{\mathrm{S},\mathcal{V}_m}(x, y) \qquad m = 0, \ldots, p-1 \tag{40}$$

where all the functions $\{\psi_m(x, y)\}_{m=0}^{p-1}$ are strictly periodic, nonvanishing and continuous, and the standard QPFs $\{F_{\mathrm{S},\mathcal{V}_m}(x, y)\}_{m=0}^{p-1}$ are strictly periodic, i.e. they all have total vorticity $N = 0$ ($N_+ = N_-$). We now distinguish between three cases. (a) Functions (40) have *no* zero in common to *all* of them. In this case, the normalized eigenvector is given simply by

$$\mathbf{V}(x, y) = \left[\sum_{m=0}^{p-1} |\Lambda_m(x, y)|^2\right]^{-1/2} \{\Lambda_m(x, y)\}_{m=0}^{p-1} \tag{41}$$

and is strictly periodic ($N = 0$) and at least continuous in (x, y). (b) Functions (40) have zeros in common to all of them, but for at least one of these zeros the vortex characteristics $(\eta, \varepsilon, \theta)$ in the case of $\Lambda_0(x, y)$ are different from those in the case of $\Lambda_m(x, y)$, for some $m > 0$. Then, using arguments similar to those leading to (35), it follows that $\tilde{v}(m; x, y)$ must have finite discontinuities at this zero. However, by considering the system of above equations satisfied by $\{\tilde{v}(m; x, y)\}_{m=1}^{p-1}$, one can easily see that such discontinuities are not possible since the elements of M are smooth functions. Therefore, this case *cannot occur*. (c) Functions (40) have precisely $\bar{N}$ zeros in common to all of them, with the *same* vortex characteristics $\{(x_{0,i}, y_{0,i}); \eta_i, \varepsilon_i, \theta_i\}_{i=1}^{\bar{N}}$. Consider the standard QPF $F_{\mathrm{S},\mathcal{V}_c}(x, y)$ with these vortex characteristics. Because of (27) and (40), $F_{\mathrm{S},\mathcal{V}_c}(x, y)$ is just the 'common factor' of

$\{\Lambda_m(x, y)\}_{m=0}^{p-1}$, so that the functions $\{\Lambda'_m(x, y) = \Lambda_m(x, y)/F_{\mathrm{S},\mathcal{V}_c}(x, y)\}_{m=0}^{p-1}$ are well defined and at least continuous. Clearly, these functions have no zero in common and they are all quasiperiodic with total vorticity $N = -N_c$, where $N_c = \sum_{i=1}^{\bar{N}} \eta_i$. By replacing $\Lambda_m(x, y)$ in (41) by $\Lambda'_m(x, y)$, we then obtain a normalized, quasiperiodic and at least continuous eigenvector with $N = -N_c$.

We remark that $N = -N_c$ is just the total vorticity of the zeros common to all the functions $\{1/\tilde{v}(m; x, y)\}_{m=1}^{p-1}$ (see (39)). Thus, by considering the phase plots of these easily calculable functions, the Chern index N is immediately determined. We also remark that a well-known relation satisfied by the Chern indices, $\sum_{l=1}^{p} N_l = 0$ (see, e.g., [4]), can be simply proved as follows. Consider the matrix S which diagonalizes M, i.e. $S^{-1}MS = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$. This matrix (and its inverse) always exists and its columns are just the orthonormal eigenvectors, $S = (\mathbf{V}_1, \ldots, \mathbf{V}_p)$. Since the eigenvector components are at least continuous functions of (x, y), also $\Delta(x, y) \equiv \det[S(x, y)]$ is such a function. In addition, it is easy to see (using, e.g., the simplified quasiperiodicity conditions (3) and (4)) that $\Delta(x, y)$ is a quasiperiodic function with total vorticity $N_T = \sum_{l=1}^{p} N_l$. Now, the vortices of a continuous function are also zeros of this function. Thus, if $N_T \neq 0$ $\Delta(x, y)$ must have zeros in $[0, 1)^2$, implying that S^{-1} does not exist for all (x, y). Therefore, $N_T = 0$.

7. Conclusions

Vortices of complex functions are singular entities associated with important physical properties and their structure can be measured experimentally. In the case of quasiperiodic functions (QPFs), the vortices form a periodic pattern and in each unit cell one generally has a finite number N_v of vortices with total vorticity N ($|N| \leqslant N_v$). While N is an important topological characterization, it is not sensitive to small changes of the system and does not provide detailed information about the QPF. A much finer characterization is given by the full vortex structure $\mathcal{V}$ in one unit cell. We have shown that a 'standard' QPF $F_{\mathrm{S},\mathcal{V}}(x, y)$ with arbitrary $\mathcal{V}$ can be explicitly constructed as a product (27) of N_v elementary QPFs associated with the individual vortices. Then, a general QPF with vortex structure $\mathcal{V}$ must be an element of the equivalence class $\mathcal{C}_{\mathcal{V}} = \{F_N(x, y) = \psi(x, y)F_{\mathrm{S},\mathcal{V}}(x, y)\}$, for all strictly periodic, nonvanishing and at least continuous functions $\psi(x, y)$. If the vortices are all positive and isotropic, $F_{\mathrm{S},\mathcal{V}}(x, y)$ is essentially the coherent-state representation of a general quantum state on a phase-space torus. This expresses the well-known fact [11] that such a state is completely determined by the $N_v = N$ vortices. In general, a QPF can be approximately reconstructed from its vortex structure $\mathcal{V}$ on the basis of $F_{\mathrm{S},\mathcal{V}}(x, y)$ and the equivalence concept (see section 5). As another application of this concept, we have proposed a simple method for calculating the continuous quasiperiodic eigenvectors of periodic matrices. The normalized eigenvectors are determined only up to an arbitrary continuous phase factor. This arbitrariness and the continuity of the eigenvectors are needed, e.g., in the construction of an orthogonal set of localized orbitals in a magnetic field ('magnetic Wannier functions') [6, 7, 10]. This is an example where a natural arbitrariness in the choice of $\psi(x, y)$ turns out to be advantageous.

The vortex-structure characterization of QPFs may have applications to the quantum-chaos problem on a phase-space torus [12–16] (see also sections 4.2 and 6). For $h = 1/p$ (p integer) and for given boundary conditions specified by a quasivector $\mathbf{w}$ in the Brillouin zone (BZ), there are p toral eigenstates $|\Psi_{l,\mathbf{w}}\rangle$, $l = 1, \ldots, p$, corresponding to p levels. As $\mathbf{w}$ varies in the BZ, each level broadens into a band l. At fixed l and in an arbitrary representation, $|\Psi_{l,\mathbf{w}}\rangle$ is generally a QPF in the BZ with Chern index (total vorticity) σ_l [13, 15]. An average increase in $|\sigma_l|$, as a nonintegrability parameter is varied, is a quantum signature of an order-chaos transition [13, 14, 16, 25]. However, since σ_l can change only at band degeneracies which are isolated in parameter space, the Chern index is *not* a sensitive measure of this transition.

Let us therefore consider the vortex structure of $|\Psi_{l,\mathbf{w}}\rangle$ at a finer level. Since $|\Psi_{l,\mathbf{w}}\rangle$ is not an analytic function of $\mathbf{w}$ in any representation, we expect it to exhibit generic vortices, i.e. anisotropic and with different signs. This vortex structure is generally much more sensitive to small variations of the parameter than the Chern index. For example, a drastic change such as the birth of a pair of vortices of opposite signs by bifurcation (compare with relation (28)), is not detected by the Chern index since σ_l is conserved under this change. Bifurcations and band degeneracies eventually generate a large number of vortices in a strong-chaos regime. An interesting question is then to what extent the vortex characteristics in this regime satisfy statistics close to those of Gaussian random waves [27]. In general, an investigation of the full vortex structure and its changes under variations of parameters should lead to a much more complete understanding of the fingerprints of classical nonintegrability on the quantum properties in a semiclassical regime.

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Appendix. Simplified quasiperiodicity conditions

We show here how the quasiperiodicity conditions (1) and (2) can be reduced to the much simpler form (3) and (4) by a suitable phase transformation. For the convenience of the reader, we also give the derivation of expression (5), first obtained in [22].

We start by deriving a relation between the total vorticity N and the functions $\alpha(x, y)$ and $\beta(x, y)$ in (1) and (2). It follows from (1) that by increasing y to $y+1$ the phase of $F(x+1, y)$ changes relative to that of $F(x, y)$ by $\alpha(x, y+1)-\alpha(x, y)$. Equation (2) implies, on the other hand, that by decreasing x from $x+1$ to x the phase of $F(x, y+1)$ changes relative to that of $F(x, y)$ by $\beta(x, y)-\beta(x+1, y)$. Thus, the total phase change $2\pi N$ by going around the unit-cell boundary counterclockwise is given by

$$\alpha(x, y+1)-\alpha(x, y)+\beta(x, y)-\beta(x+1, y)=2\pi N. \tag{42}$$

The phase $\alpha(x, y)$ in (1) can be eliminated by making a phase transformation $F(x, y) \rightarrow F'(x, y)=\exp[-\mathrm{i}\varphi'(x, y)]F(x, y)$ so that

$$F'(x+1, y)=F'(x, y). \tag{43}$$

This is accomplished by requiring that

$$\varphi'(x+1, y)-\varphi'(x, y)=\alpha(x, y). \tag{44}$$

If $\alpha(x, y)$ is strictly periodic in x ($\alpha(x+1, y)=\alpha(x, y)$), equation (44) is satisfied by choosing $\varphi'(x, y)=x\alpha(x, y)$. Otherwise, we assume, without loss of generality, that the smooth function $\alpha(x, y)$ is a polynomial of arbitrary finite order L in x:

$$\alpha(x, y)=\sum_{l=0}^{L}\bar{\alpha}_l(y)x^l. \tag{45}$$

Accordingly, we assume that

$$\varphi'(x, y) = \sum_{l=1}^{L+1} \bar{\varphi}_l(y) x^l. \tag{46}$$

By inserting (45) and (46) in (44), we find that the unknown coefficients $\bar{\varphi}_l(y)$ satisfy the linear system of equations

$$\sum_{l=l'+1}^{L+1} \binom{l}{l'} \bar{\varphi}_l(y) = \bar{\alpha}_{l'}(y) \qquad l' = 0, \ldots, L \tag{47}$$

where $\binom{l}{l'}$ are binomial coefficients. The system (47) can be easily solved by iteration, starting from $l' = L$: $\bar{\varphi}_{L+1}(y) = \bar{\alpha}_L(y)/(L+1)$, $\bar{\varphi}_L(y) = \bar{\alpha}_{L-1}(y)/L - \bar{\alpha}_L(y)/2$, etc.

In the y variable, $F'(x, y)$ satisfies the quasiperiodicity condition:

$$F'(x, y+1) = \exp[\mathrm{i}\beta'(x, y)]F'(x, y) \tag{48}$$

where $\beta'(x, y) = \beta(x, y) + \varphi'(x, y) - \varphi'(x, y+1)$. Using (42) and (44), we find that

$$\beta'(x, y) - \beta'(x+1, y) = 2\pi N. \tag{49}$$

Then, by writing $\beta'(x, y) = \tilde{\beta}(x, y) - 2\pi N x$, it follows from (49) that $\tilde{\beta}(x, y)$ is strictly periodic in x, $\tilde{\beta}(x+1, y) = \tilde{\beta}(x, y)$. If the phase $\tilde{\beta}(x, y)$ could be eliminated by a second-phase transformation, $F'(x, y) \to F_N(x, y) = \exp[-\mathrm{i}\tilde{\varphi}(x, y)]F'(x, y)$, $F_N(x, y)$ would satisfy, because of (48), condition (4). To eliminate $\tilde{\beta}(x, y)$, we require that

$$\tilde{\varphi}(x, y+1) - \tilde{\varphi}(x, y) = \tilde{\beta}(x, y). \tag{50}$$

Equation (50) can be solved for $\tilde{\varphi}(x, y)$ by the same procedure used above to solve equation (44). Since $\tilde{\beta}(x+1, y) = \tilde{\beta}(x, y)$, also $\tilde{\varphi}(x+1, y) = \tilde{\varphi}(x, y)$. Thus, $F_N(x, y)$ is strictly periodic in x like $F'(x, y)$ (see (43)), i.e. it satisfies also condition (3).

Expression (5) for $N \neq 0$ is derived as follows. Since $F_N(x, y)$ is periodic in x (equation (3)), it can be expanded in a Fourier series with coefficients $f_n(y)$, n integer. Using this expansion in equation (4), we find that $f_n(y+1) = f_{n+N}(y)$. By iterating this relation forward and backward l times starting from any of the $|N|$ initial values $n = s = 0, \ldots, |N|-1$, it follows that $f_n(y) = f_s(y+l)$, where n is uniquely decomposed as $m = lN + s$. Thus, for all n, $f_n(y)$ can be simply expressed in terms of just the $|N|$ functions $\{f_s(y)\}_{s=0}^{|N|-1}$. In turn, these latter functions can always be written as finite Fourier expansions $f_s(y) = \sum_{j=0}^{|N|-1} \exp(2\pi\mathrm{i}js/|N|)\phi_j(y + s/N)$, where $\phi_j(y) \equiv |N|^{-1} \sum_{s=0}^{|N|-1} \exp(-2\pi\mathrm{i}js/|N|) f_s(y - s/N)$, $j = 0, \ldots, |N|-1$. Expression (5) is then obtained.

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