General Relativity and Gravitation, Vol. 13, No. 8, 1981

Retardation in Rigid Motion and Degree of Rigidity

ITZHAK DANA

Department of Physics, Technion–Israel Institute of Technology, Haifa, Israel

Received October 15, 1980

Abstract

Retardation features of rigid motion, in flat two-dimensional space-time, are interpreted in terms of a consistent picture of signals. A similar analysis is adopted to effect a natural transition from rigid to nonrigid frames for which a degree of rigidity, having a simple physical meaning, can be defined.

There are several ways to define and characterize rigid frames in relativity [1-4]. In this paper we present a new approach to the problem, restricting ourselves to the case of flat two-dimensional space-time. The analysis proposed has two main advantages: (1) it reveals retardation features in the transformation to rigid frames, which can be interpreted by a consistent picture of signals; (2) it can be used to effect a continuous transition from rigid to nonrigid frames, and to define a dimensionless degree of rigidity [5, 6].

Let R be a frame of reference which accelerates in the direction of the X axis of an inertial system I. If R is rigid, its points of reference can be represented [3] by a Cartesian system of coordinates (x, y, z) with origin at O and with its x axis parallel to the X axis of I. Let t be the time coordinate of R and T the standard time in I. The transformation from I to R

$$X = \psi(x, t), \qquad T = t, \tag{1}$$

satisfies the equation [3]

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2 = 1 \tag{2}$$

It can be shown, by standard procedures [7], that equations (1) and (2) lead to a Euclidean spatial geometry in $R, dl^2 = dx^2 + dy^2 + dz^2$. The equation for the velocity of the point of reference (x, y, z) relative to I, $v(x, y, z, T) = V(x, t)$, can be found from equation (2), or intuitively [4, 8]

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial t}\left(1 - \frac{V^2}{c^2}\right)^{1/2} \tag{3}$$

The general solution of equation (3) with boundary conditions on the t axis, $V(0, t) = V_0(t)$, satisfies the equation [4, 8]

$$V(x, t) = V_0\left\{t - \frac{x}{c^2} V(x, t)\left[1 - \frac{V^2(x, t)}{c^2}\right]^{-1/2}\right\} \tag{4}$$

where V_0 is a function of the expression inside the curly brackets, and we have [3, 8]

$$X = \psi(x, t) = \int^t V_0(t')\,dt' + \int_0^x \left[1 - \frac{V^2(x', t)}{c^2}\right]^{1/2} dx' \tag{5}$$

Relation (4) was used by the author [8] to define, for any rigid frame R, a set of inertial systems of reference I', with velocities in the range of $V_0(t)$. By introducing a suitable mapping between events in R and events in each element of this set, nonsimultaneous events in R will correspond to simultaneous events in I'. Here we shall reveal retardation features of equation (4), which can be interpreted by a consistent picture of signals. These retardation phenomena are also noted when equation (4) is explicitly solved for some V_0's [8].

Let us write equation (4) in the more general form

$$V(x_2, t_2) = V(x_1, t_1) \tag{6}$$

$$t_2 = t_1 + \frac{x_2 - x_1}{c^2} V(x_1, t_1)\left[1 - \frac{V^2(x_1, t_1)}{c^2}\right]^{-1/2} \tag{7}$$

It is possible to interpret equations (6) and (7) as follows: the information that the plane $x = x_1$ has a velocity $V = V(x_1, t_1)$ relative to I, is carried by a signal which propagates along the x axis of R. It is transmitted from x_1 at time t_1 and when it arrives at x_2 at time t_2 [assuming, without loss of generality, that $t_2 > t_1$ in equation (7)], the velocity of x_2 is adjusted to $V(x_2, t_2) = V$. This adjustment is continuous, since signals arrive at x_2 at any time, and from all x_1's for which $t_2 > t_1$ in equation (7).

We have thus associated with R an infinite bundle of signals, each carrying information about a particular velocity V in the range of $V(x, t)$ [which is the same range for all x, because of equation (4)]. The phase velocity $u(V)$ of such a signal, relative to I, can be calculated from equations (5)-(7), assuming an infinitesimal value dx for $x_2 - x_1$. We obtain the simple result

$$u(V) = c^2/V \tag{8}$$

This picture is consistent in the sense that when a signal with phase velocity $u(V)$ arrives at $x_2 = x_1 + dx$ at time $t_2 = t_1 + dt, x_2$ assumes the velocity V relative to I, and we can look at x_2 as the new source of the signal.

There exists an interesting physical analog to this picture. Let the points of reference of R be represented by particles of mass m, momentum p, and energy $E = (p^2c^2 + m^2c^4)^{1/2}$. Since R is rigid, these particles perform Born rigid motions [1] under the influence of external forces. If these forces are simultaneously switched off at time T_0, the particles will move at constant velocities, relative to I. For $T > T_0$, they will define an inertial system $I(T_0)$, characterized by a time-dependent spatial geometry. We now associate with each particle [point of reference of $I(T_0)$] its de Broglie wave and consider the wave front lying at the position of the particle at $T = T_0$. The phase velocity of this wave front is

$$u_{\text{ph}} = \frac{E}{p} = \frac{c^2}{V} \tag{9}$$

where V is the constant velocity of the particle considered, relative to I. From a comparison of equations (9) and (8), we conclude that the wave front follows the actual position of the points of reference of R, which have the velocity V relative to I. We can say that the de Broglie waves associated with $I(T_0)$ contain all the information about the rigid motion of R, for $T > T_0$. This suggests that rigid motion is closely related to the time evolution of a free particle wave function.

The picture of signals just described cannot depend on the particular system I, from which we look at R. This means that under a Lorentz transformation $I \to \bar{I}$, where $\bar{I}$ moves relative to I at a velocity v, we must have

$$u(\bar{V}) = u\left(\frac{V + v}{1 + Vv/c^2}\right) = \frac{u(V) + v}{1 + u(V)v/c^2} \tag{10}$$

where $u(V)$ $[u(\bar{V})]$ is the phase velocity, relative to I $(\bar{I})$, of the signal carrying information about the velocity $V(\bar{V})$. It is easily checked that the universal function (8) satisfies equation (10).

The other solutions of equation (10) correspond to the phase velocities of signals associated to nonrigid frames $\tilde{R}$. The interpretation of the motion of such a frame, in terms of its bundle of signals, should not depend on the particular inertial system I from which we look at it. The general form of the transformation from I to $\tilde{R}$, with the choice $t = T$ for the time coordinate t of $\tilde{R}$, will now be derived.

The functional relation (10) admits the general solution

$$u_r(V) = c\,\frac{r + V/c}{1 + rV/c} \tag{11}$$

where r is an arbitrary parameter which characterizes a particular function $u_r(V)$. When $|r| = \infty$, we obtain the solution (8), which corresponds to the rigid frame R.

The most general transformation in flat two-dimensional space-time, with $T = t$, can always be written in the form [analogous to the transformation (5)]

$$X = \int^t V_0(t')\,dt' + \int_0^x \gamma(x', t)\,dx' \tag{12}$$

For each value of the parameter r in equation (11), a function $\gamma_r(x, t)$ in equation (12) has to be found, such that the motion of the corresponding frame $\tilde{R}_r$ can be interpreted in terms of signals with phase velocities $u_r(V)$.

From equation (12) we see that the length of a segment dx, relative to I, is given by $\gamma_r(x, t)\,dx$. For the velocity $V(x, t)$ we then obtain the equation [analogous to equation (3) for the rigid frame]

$$\frac{\partial V}{\partial x} = \frac{\partial \gamma_r}{\partial t} \tag{13}$$

If the point of reference $x_2 = x_1 + dx$ assumes the velocity $V = V(x_1, t_1)$ after a time dt, and the retardation is due to the propagation of a signal with phase velocity $u_r(V)$ relative to I, we must have

$$u_r(V) = \gamma_r \left.\frac{\partial x}{\partial t}\right|_V + V \tag{14}$$

Using equations (11), (13), (14) and

$$\left.\frac{\partial x}{\partial t}\right|_V = -\frac{\partial V}{\partial t}\bigg/\frac{\partial V}{\partial x}$$

a differential equation for $\gamma_r(x, t)$ is obtained, with the general solution

$$\gamma_r(x, t) = S(x)\left(1 - \frac{V^2}{c^2}\right)^{1/2}\left(\frac{1 - V/c}{1 + V/c}\right)^{1/2r} \tag{15a}$$

for $r \neq 0$, and

$$\gamma_0(x, t) = \beta(x) + \gamma(x)t \tag{15b}$$

where S, β, and γ are arbitrary functions of x, and $\tilde{R}_0$ is essentially an inertial frame. The coordinate x can be redefined so that $S(x) = 1$. If we then substitute

$\gamma_r(x, t)$ in equation (13) we obtain a differential equation which can in principle be solved for $V(x, t)$ when boundary conditions on the t axis are given.

The transition from nonrigid to rigid frames can be continuously effected, by letting $|r| \to \infty$ in equation (15a). On the other hand, when $\tilde{r} = |r|^{-1}$ departs from zero, we do the opposite transition. The parameter r has a simple physical significance: according to equation (11), $c|r|$ is the (phase) velocity of the wave front of a signal, relative to the point of reference on which it momentarily lies. It seems then natural to identify the quantity $|r|$ as the degree of rigidity of the frame $\tilde{R}_r$. Accordingly, the frames $\tilde{R}_r$ and $\tilde{R}_{-r}$ assume the same degree of rigidity (as the case is for the rigid frame, when $\tilde{R}_\infty$ and $\tilde{R}_{-\infty}$ coincide).

In previous works [5, 6] the problem of rotatory motion with a maximum degree of rigidity was considered. It may be interesting to generalize the analysis of rigidity in terms of signals to include such cases of nonrectilinear motion, or transformations in curved space-time.

References

1. Born, M. (1909). *Ann. Phys. (Leipzig)*, **30**, 1.
2. Rosen, N. (1947). *Phys. Rev.*, **71**, 54.
3. Newman, E. T., and Janis, A. I. (1959). *Phys. Rev.*, **116**, 1610.
4. Cavalleri, G., and Spinelli, G. (1970). *Nuovo Cimento*, **66B**, 11.
5. Cavalleri, G. (1968). *Nuovo Cimento*, **53B**, 415.
6. Newburgh, R. G. (1974). *Nuovo Cimento*, **23B**, 365.
7. Møller, C. (1972). *The Theory of Relativity*, Clarendon Press, Oxford.
8. Dana, I. (1979). TECHNION-PH-79-45, preprint.