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Quantum chaos on a toral phase space for general boundary conditions: recent new results

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Abstract

This paper is a brief review of recent new results for the quantum-chaos problem on a 2D toral phase space for general boundary conditions (BCs) on the torus. The following results will be reviewed and discussed: (a) The new concept of *band distribution* for Hamiltonians periodic in phase space. (b) Expressions for the *general* smooth torus map and for the condition determining the allowed quantum BCs associated with this map. (c) An exact renormalization scheme for quantum Anosov maps on the torus, which "eliminates" the general-BCs problem, replacing it by a fixed-BCs one. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Nonintegrable systems whose dynamics can be reduced to a 2D torus in phase space have attracted much attention in the quantum-chaos literature [1,2]. Several such systems, e.g., the paradigmatic kicked rotor [3–9] and kicked Harper model [10–22], have been considered in most studies on a cylindrical phase space, where the classical chaotic diffusion in the unbounded phase-space direction leaves interesting fingerprints in quantum-chaotic phenomena such as dynamical localization [3–9] (in the kicked rotor) and quantum diffusion [10–12] (in the kicked Harper model). The framework of a toral phase is useful and interesting because of two main reasons. First, this phase space is compact, with a *well-defined finite* number p of independent quantum states for given boundary conditions (BCs) on the torus, where $p \propto 1/\hbar$ and $\hbar$ denotes here a dimensionless (scaled) Planck's constant. The compactness of the phase space is very convenient for studying basic aspects of the classical-quantum correspondence in nonintegrable systems [5,13–41] and allows to derive several exact results, especially for the quantum cat maps [27–35].

Second, interesting and essentially unexplored quantum-chaotic phenomena have manifestations in

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the sensitivity of the eigenstates to variations in the toral BCs, specified by a 2D Bloch wave vector $\boldsymbol{w}$ taking values in the "dual" torus, a "Brillouin zone" (BZ). For example, in the special case of Hamiltonians periodic in phase space, e.g., the kicked Harper model, it is numerically observed that a band of eigenstates concentrated closer to the classical chaotic region exhibits higher sensitivity to variations of $\boldsymbol{w}$ [14,22]. A topological quantity associated with the band, the Chern index [14,15,20–22], is only a *weak* measure of this sensitivity [22]. It has also been realized [21,22] that, since $\boldsymbol{w}$ is of a purely quantum nature, quantities associated with individual eigenstates should be averaged over the BCs in order to approach correctly the corresponding classical quantity in the semiclassical limit. Thus, while most studies have been confined to special BCs (mainly the strictly periodic BCs with $\boldsymbol{w}=\boldsymbol{0}$), it is now well recognized that *all* the BCs must be taken into account in order to obtain a complete understanding of the fingerprints of classical nonintegrability in the *full* quantum dynamics.

Very recently, some progress has been made in the study of the quantum-chaos problem on a toral phase space for general BCs. The purpose of this paper is to review and discuss briefly the new results. In Section 2, which serves only as a general background, we summarize the case of Hamiltonians periodic in phase space [14,15,20,23]. The new concept of *band distribution* [21,22] for these Hamiltonians is reviewed in Section 3. In Section 4, we review the recent results [40] concerning the *general* smooth torus map and the allowed quantum BCs associated with it. Finally, in Section 5, we briefly summarize the exact renormalization scheme developed quite recently [42] for quantum Anosov maps on the torus, which "eliminates" the general-BCs problem, replacing it by a *fixed*-BCs one.

2. Periodic Hamiltonians in phase space

Consider a Hamiltonian $H(u,v;t)$ periodic in the phase space (u,v) with a $2\pi\times 2\pi$ unit cell, which is the basic torus T^2. A well-known example is the time-independent integrable Harper model [43–49], describing the problem of Bloch electrons in a magnetic field in a one-band or one-Landau-level approximation; corrections to this model including many bands or Landau levels are known [50]. The simplest nonintegrable example is the kicked Harper model [10–22], which is exactly related [18,19] to the realistic system of periodically kicked charges in a uniform magnetic field under resonance conditions [51,52]. Assuming, in general, a time-periodic Hamiltonian, the "quantum map" is the evolution operator $\hat{U}(\hat{u},\hat{v})$ in one time period, where $[\hat{u},\hat{v}]=\mathrm{i}\hbar$. If $\rho=\hbar/2\pi$ is a rational number, $\rho=q/p$ (q and p are coprime integers), there exists a pair of "smallest" phase-space translations $\hat{D}_1=\mathrm{e}^{\mathrm{i}\hat{u}/\rho}$ and $\hat{D}_2=\mathrm{e}^{-\mathrm{i}p\hat{v}}$ commuting with $\hat{U}$ and among themselves. Clearly, $\hat{D}_1$ ($\hat{D}_2$) is a translation by 2π ($2\pi q$) in the v (u) direction, defining the "quantum" toral phase space T^2_{Q}: $0\leqslant u<2\pi q$, $0\leqslant v<2\pi$, with $T^2_{\mathrm{Q}}=qT^2$. The general quantum states in this phase space are the simultaneous eigenstates $|\Psi_{\boldsymbol{w}}\rangle$ of $\hat{D}_1$ and $\hat{D}_2$, where $\boldsymbol{w}$ is a Bloch wave vector with components (w_1,w_2) labeling the eigenvalues $\mathrm{e}^{\mathrm{i}w_1/\rho}$ and $\mathrm{e}^{-\mathrm{i}pw_2}$ of $\hat{D}_1$ and $\hat{D}_2$, respectively, and taking all values in the "Brillouin zone" (BZ) torus T_{BZ}: $0\leqslant w_1<2\pi\rho$, $0\leqslant w_2<2\pi/p$. The eigenvalue equations for $\hat{D}_1$ and $\hat{D}_2$ define, for each $\boldsymbol{w}$ in T_{BZ}, *boundary conditions* (BCs) in T^2_{Q}. At fixed BCs, there are precisely p eigenstates $|\Psi_{b,\boldsymbol{w}}\rangle$ of $\hat{U}$ with quasienergy levels $\omega_b(\boldsymbol{w})$, where $b=1,\ldots,p$ is a "band" index.

Assuming the nondegeneracy of $\omega_b(\boldsymbol{w})$ for all $\boldsymbol{w}$, the total phase change of $|\Psi_{b,\boldsymbol{w}}\rangle$ by going around the boundary of T_{BZ} counterclockwise is given by $2\pi\sigma_b$, where σ_b is a topological integer, the *Chern index* [14,15,20,22]. This is completely analogous to the integer Hall conductance carried by a magnetic band in a perfect crystal [44–49]. The eigenstates $|\Psi_{b,\boldsymbol{w}}\rangle$ may be weakly dependent on $\boldsymbol{w}$ *only* if $\sigma_b=0$, a value which may arise only if $q=1$ [20,47]. In this case, where $T^2_{\mathrm{Q}}=T^2$, one can easily establish a classical-quantum correspondence on the torus for small $\hbar$ [14,15,27]. Several arguments [14,15,53], supported by numerical evidence, indicate that if the Husimi distribution of an eigenstate is well localized on a stability island the corresponding band has $\sigma_b=0$. On the other hand, eigenstates whose Husimi distribution is spread over the classical chaotic region should belong to bands with $\sigma_b\neq 0$. The transition from a nearly integrable regime, where almost all $\sigma_b=0$, to a fully chaotic regime, where almost all $\sigma_b\neq 0$, as a nonintegrability parameter is increased, takes place via degeneracies between adjacent bands, leading to a "diffusion" of the Chern indices [15,54]. The last three sentences summarize what we call the Chern-index characterization of the classical-quantum correspondence on the torus.

3. Band distributions

There is a sense in which eigenstates are not natural for a characterization of the classical-quantum correspondence on the torus, in that they exhibit rather nonclassical features: (a) They are associated with the purely quantum quantity $\boldsymbol{w}$ [the "quasicoordinate" of (u, v), by definition] and they are generically sensitive to this quantity. (b) In the general case $q \neq 1$, an eigenstate may be viewed as arising from quantum tunneling between q degenerate classical orbits located in the q adjacent tori T^2 composing T^2_{Q} [15,16]. As a consequence, σ_b is always nonzero [15,20] and the Chern-index characterization above cannot be extended to this case. In Refs. [21,22], we have proposed a remedy to all this, the new concept of *band distribution* (BD):

$$P_b(u, v) = \frac{1}{|T_{\mathrm{BZ}}|} \int_{T_{\mathrm{BZ}}} \mathrm{d}\boldsymbol{w}\, P_{b,\boldsymbol{w}}(u, v), \tag{1}$$

where $P_{b,\boldsymbol{w}}(u, v)$ is an eigenstate phase-space probability distribution (Wigner or Husimi) and $|T_{\mathrm{BZ}}|$ is the area of T_{BZ}. The BD may be viewed as the representative distribution for level b in the torus and is associated with a mixed state, i.e., the density matrix $|T_{\mathrm{BZ}}|^{-1} \int_{T_{\mathrm{BZ}}} \mathrm{d}\boldsymbol{w}\, |\Psi_{b,\boldsymbol{w}}\rangle\langle\Psi_{b,\boldsymbol{w}}|$ for band b. We have shown that the BD has several remarkable properties:

(A) *All* the eigenstate distributions $P_{b,\boldsymbol{w}}(u, v)$ can be exactly reproduced from the BD. For example, in the Wigner case, one has

$$P_{b,\boldsymbol{w}}(u, v) = \sum_{r,s=-\infty}^{\infty} A_b(r, s; \boldsymbol{w}) \delta(u - w_1 - r\pi\rho) \delta(v - w_2 - s\pi/p) \tag{2}$$

and we find the following formula for the coefficients $A_b(r, s; \boldsymbol{w})$ in terms of the Wigner BD [21]:

$$A_b(r, s; \boldsymbol{w}) = \frac{\pi^2 q}{p^2} \sum_{k,l=0,1} (-1)^{ks+lr+klp} \times P_b(u + kq\pi,\ v + l\pi), \tag{3}$$

where $u = w_1 + r\pi\rho$ and $v = w_2 + s\pi/p$. In the Husimi case, the expression for $P_{b,\boldsymbol{w}}(u, v)$ in terms of the BD is more complicated [55] and can be obtained from Eqs. (2) and (3) using the fact that the Husimi distribution is a Gaussian smoothing of the Wigner one. Now, the Husimi $P_{b,\boldsymbol{w}}(u, v)$ assumes exactly p zeros in T^2_{Q} [22,23], which determine completely the eigenstate $|\Psi_{b,\boldsymbol{w}}\rangle$ itself (more precisely, its coherent-state representation) [23]. Thus, *no information is lost* about the individual eigenstates by the averaging in Eq. (1). This result is somehow surprising, since Eq. (1) is analogous to the usual definition of a Wannier function (the uniform average of the Bloch functions in a band), and it is well known [47,48] that phase-space Bloch functions can be fully recovered from the corresponding Wannier function only if $\sigma_b = 0$. On the other hand, definition (1) holds for *arbitrary* σ_b. This may be understood [55] by using a more general definition of phase-space Wannier functions for arbitrary σ_b [49,56].

(B) $P_b(u, v)$ is periodic with unit cell T^2 (the classical torus) for *general* q, in contrast with $P_{b,\boldsymbol{w}}(u, v)$, whose unit cell of periodicity is, generally, T^2_{Q}. The BD is then analogous to a classical probability distribution in the phase space T^2.

(C) In the Husimi case, the BD *never vanishes* [$P_b(u, v) > 0$ in T^2], simply because the p zeros of $P_{b,\boldsymbol{w}}(u, v)$ generally vary with $\boldsymbol{w}$ and definition (1) involves an integration over all $\boldsymbol{w}$. The Husimi zeros are rather "nonclassical", for example, they do not allow $P_{b,\boldsymbol{w}}(u, v)$ to approach, in the semiclassical limit, the microcanonical uniform distribution in a strong-chaos regime [14,23].

We thus see that *all* the quantum information about the individual eigenstates is encoded in, and *fully* recoverable from the BD, and, at the same time, the BD is closer to a classical distribution than the eigenstate distributions. This should make the BDs most suitable to study the classical-quantum correspondence on the torus. One may generalize the BD concept by averaging over a set of N adjacent bands $b = b_1, \ldots, b_N$. This set may be considered as a single entity, a *generalized band* (GB), which can be characterized by its total Chern index, $\sigma_{\mathrm{GB}} \equiv \sum_{b=b_1}^{b_N} \sigma_b$, and by the *generalized* BD (GBD) associated with it, $P_{\mathrm{GB}}(u,v) = N^{-1}\sum_{b=b_1}^{b_N} P_b(u,v)$. The further averaging over bands should give a "more classical" BD, as when smoothing over many levels in a general quantum system [57]. Important properties of GBDs are:

(D) The GBD for two bands is approximately conserved when a parameter is varied across a degeneracy point, despite the fact that the separate BDs may change drastically [22].

(E) Let us assume that the generator $\hat{G}$ of $\hat{U} = \mathrm{e}^{-\mathrm{i}\hat{G}}$ belongs to the class of torus Hamiltonians to which the renormalization-group approach in Ref. [49] is applicable. Then, if $\rho' = q'/p'$ is sufficiently close to $\rho = 1/p$, $p' \gg p$, the p' bands for ρ' can be grouped into p "clusters" or GBs C_b of adjacent bands, such that: (a) The quasienergy interval covered by the bands in C_b is relatively close to that covered by band b. (b) The total Chern index $\sigma(C_b)$ of C_b is equal to σ_b. (c) The GBD $P'_{C_b}(u',v')$ is approximately equal to $P_b(u,v)$ and $P'_{C_b}(u',v') \to P_b(u,v)$ as $\rho' \to \rho$. Thus, if $P_b(u,v)$ is concentrated on a stability island $[\sigma_b = \sigma(C_b) = 0]$ or on the chaotic region $[\sigma_b = \sigma(C_b) \neq 0]$, the same will be true for the GBD for C_b. In this way, the Chern-index characterization of the classical-quantum correspondence is extended to cases of $q \neq 1$.

4. The general torus map

In general, the Hamiltonian of a system whose dynamics can be reduced to a torus is *not* periodic in phase space. Well-known examples are the standard map, derived from the kicked-rotor Hamiltonian [3–9], and the cat map [27–35], whose Hamiltonian is quadratic in the phase-space variables [30]. It is natural to ask what is the most general form of an area-preserving map on a 2D torus and how to quantize it properly. These questions have been considered quite recently by Keating, Mezzadri, and Robbins (KMR) [40] for smooth systems (thus, for example, the discontinuous baker map [2] is excluded). First, a smooth map M on T^2 can be expressed uniquely as the composition of two maps, $M = M_A \circ M_1$. Here M_A is a cat map, $M_A(\boldsymbol{z}) = A \cdot \boldsymbol{z} \bmod 2\pi$, where $\boldsymbol{z}$ is the column vector $(u,v)^{\mathrm{T}}$ and A is a 2×2 integer matrix with $\det(A) = 1$. The map M_1 is defined by $M_1(\boldsymbol{z}) = \boldsymbol{z} + \boldsymbol{F}(\boldsymbol{z}) \bmod 2\pi$, where $\boldsymbol{F}(\boldsymbol{z})$ is a 2π-periodic vector function of $\boldsymbol{z}$. The quantization of $M = M_A \circ M_1$, with $\hbar = 2\pi/p$ $(q = 1)$, is then the unitary operator $\hat{U} = \hat{U}_A\hat{U}_1$, where $\hat{U}_A$ is the "quantum cat map" [27,30] and $\hat{U}_1$ is the quantization of M_1. If one assumes that M_1 is the map for a periodic Hamiltonian with unit cell T^2 (this is the case if and only if $\int_{T^2} \boldsymbol{F}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = \boldsymbol{0}$ [40]), $\hat{U}_1$ is the one-step evolution operator for the Weyl quantization of this Hamiltonian and is a periodic operator function representable by a well-defined Fourier expansion.

Under the assumption above, KMR show that an eigenstate $|\Psi_{\boldsymbol{w}}\rangle$ of $\hat{D}_1$ and $\hat{D}_2$ can be an eigenstate of $\hat{U}$ only for those values of $\boldsymbol{w}$ satisfying

$$A \cdot \boldsymbol{w} = \boldsymbol{w} + \pi\boldsymbol{y} \bmod 2\pi/p, \tag{4}$$

where $\boldsymbol{y} \equiv (A_{1,1}A_{1,2},\ A_{2,1}A_{2,2})^{\mathrm{T}}$. Apart from the constant $\pi\boldsymbol{y}$ (in many cases $\pi\boldsymbol{y} = \boldsymbol{0} \bmod 2\pi/p$), relation (4) means that the allowed toral BCs are given by the fixed points of A in T_{BZ}. In the case of periodic Hamiltonians, $A = I$ (the identity) and $\boldsymbol{y} = \boldsymbol{0}$, so that Eq. (4) is trivially satisfied by any $\boldsymbol{w}$, i.e., all $\boldsymbol{w}$'s in T_{BZ} are allowed, as we already know (see Section 2). For kicked-rotor Hamiltonians, with $A_{1,1} = A_{1,2} = A_{2,2} = 1$ and $A_{2,1} = 0$, the allowed $\boldsymbol{w}$'s form a continuous 1D segment in T_{BZ} [55]. The most interesting case, to be considered in more detail in the next section, is that of the perturbed hyperbolic cat maps, defined by $|\mathrm{Tr}(A)| > 2$ [36–40]. These maps, to which we shall refer, in what follows, as Anosov maps, may be viewed as *generic* torus maps in the sense that $|\mathrm{Tr}(A)| > 2$ is generically satisfied by a randomly chosen A. Quantum Anosov maps (QAMs) were originally introduced [36] to show that typical spectral properties, fitting generic eigenvalue statistics, are already found by slightly perturbing [i.e., with small $\boldsymbol{F}(\boldsymbol{z})$] the quantum cat maps whose spectra are highly degenerate [27,30] and thus atypical. According to Anosov's theorem [58], slightly perturbed cat maps have essentially the same classical dynamics, in particular they are purely chaotic, as the unperturbed cat maps. This ceases to be the case for larger perturbations that cause bifurcations generating elliptic islands [39]. However, according to Eq. (4), the BCs for QAMs are the same as those for the corresponding quantum cat map, independently of the size of the perturbation. As a matter of fact, relation (4) for the quantum cat maps was first derived by Knabe [29] several years ago.

5. Exact renormalization of QAMs

In the case of QAMs, i.e., $|\mathrm{Tr}(A)| > 2$, there are precisely $k = |2-\mathrm{Tr}(A)|$ distinct solutions $\boldsymbol{w}$ of Eq. (4), as the number of fixed points of M_A [30], forming a *finite lattice* in T_{BZ}. This is an important property of QAMs, which forms the basis for an exact renormalization scheme introduced quite recently [42]. This scheme "eliminates" in practice, the general-BCs problem, replacing it by a *fixed*-BCs one. More specifically (see more details in Ref. [42]), we define first a renormalization operator $\mathscr{R}$ whose repeated application on a QAM $\hat{U}$ generates a sequence of QAMs $\hat{U}^{(n)} = \mathscr{R}^n(\hat{U})$ on the same torus T^2. The number of BCs for $\hat{U}^{(n)}$ is k for all n, and $\hat{U}^{(n)}$ is associated with a renormalized Planck's constant $\hbar^{(n)} = \hbar/k^n$. Thus, $\hat{U}^{(n)}$ has $p^{(n)} = k^n p$ eigenstates at fixed BCs. For $\mathrm{Tr}(A) < -2$, one can always ensure that the quantum cat maps are fixed points of $\mathscr{R}$, i.e., $\hat{U}_A$ and $\mathscr{R}(\hat{U}_A)$ are the same except for the scaling in the Planck's constant. For $\mathrm{Tr}(A) > 2$, this is possible if and only if there exists an integer matrix E satisfying $\det(E) = -1$ and $[A, E] = 0$. If such a matrix does not exist, one can still define $\mathscr{R}$ so that $\mathscr{R}(\hat{U}_A)$ is associated with the matrix A' obtained from A by exchanging both its diagonal and off-diagonal elements. This implies that $\hat{U}_A$ is a fixed point of $\mathscr{R}^2$. Thus, general $\hat{U}^{(n)}$ or $\hat{U}^{(2n)}$ represent perturbations of a given quantum cat map in its classical limit $n \to \infty$.

We then show that the quasienergy eigenvalue problem for $\hat{U}^{(n)}$ for *all* k BCs is equivalent, by a unitary transformation accompanied by a scaling of the phase-space variables, to that for $\hat{U}^{(n+1)}$ at some *fixed* BCs. The latter correspond, for $n = 0$ ($\hat{U}^{(0)} = \hat{U}$), to one of the four symmetry points in T_{BZ}, depending on A. For $n > 0$, however, the fixed BCs can be only of two types, i.e., strict *periodicity* ($\boldsymbol{w}^{(n+1)} = \boldsymbol{0}$) for kp even and *antiperiodicity* [$\boldsymbol{w}^{(n+1)} = (\pi, \pi)^{\mathrm{T}}/p^{(n+1)}$] for kp odd. Thus, the total (all BCs) spectrum of $\hat{U}^{(n)}$, $n = 0, \ldots, \bar{n} - 1$ ($\bar{n} > 1$), coincides with a fraction $k^{1+n-\bar{n}}$ of the spectrum of $\hat{U}^{(\bar{n})}$ for one of these two types of BCs, and the corresponding eigenstates are related by the transformation above. In particular, the total spectrum of a quantum cat map for $\hbar = 2\pi/p$ coincides with a fraction k^{1-n} of its fixed-BCs spectrum for $\hbar = 2\pi/(k^n p)$, with arbitrary or even $n > 0$. It is interesting to note that the two types of BCs above have been studied in detail in the literature [27–39] ($\boldsymbol{w} = \boldsymbol{0}$ in almost all studies; antiperiodic BCs have been considered in Ref. [34]), so that several results turn out now to be relevant also for general BCs.

In conclusion, the significance of this scheme is in that it sheds a new light on the nature of the general toral BCs: The *full* quantum dynamics for the k BCs contains *all* the information about a special-BCs quantum dynamics at the *smaller* Planck's constant $\hbar' = \hbar/k$. The k BCs are "mixed" by the latter quantum dynamics. As k increases, $\hbar'$ tends to the classical limit. Actually, k increases exponentially with the Lyapunov exponent γ for A. In turn, γ can be increased linearly with some integer s by simply replacing M by the map M^s, which has essentially the same phase-space structure (e.g., the same island structure for large perturbations [39]) as that of M.

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