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Periodic orbits and chaotic-diffusion probability distributions

Itzhack Dana, Vladislav E. Chernov[1]

Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

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Abstract

Periodic-orbit (PO) formulas for chaotic-diffusion probability distributions (PDs) are examined in the case of the perturbed Arnol'd cat map on the cylinder. This translationally invariant system exhibits a transition from uniform to nonuniform hyperbolicity as the perturbation parameter is increased. Two *coarse-grained* PDs, describing the "diffusion" between unit cells of the system, are studied: (a) a PD based on PO ensembles; (b) a PD based on generic ensembles. The approximate PO formula for PD (b) gives results which fluctuate around the expected Gaussian distribution for all parameters considered and thus agree qualitatively with results from standard methods. The exact PO formula for PD (a) gives similar results only for sufficiently small parameters. The results for large parameters decrease monotonically relative to the Gaussian distribution. This deviation seems to disappear as the PO period is increased.

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1. Introduction

Deterministic chaos in generic Hamiltonian systems is a dynamical behavior much more complex than a purely random motion. This is mainly due to the very intricate mixture of chaotic and regular motions on all scales of phase space, leading to quasiregularity of chaos and to slow (power-law) decay of correlations [1–5]. However, strongly or completely chaotic systems may exhibit properties quite similar to those

[1] Permanent address: Mathematical-Physics Department, Voronezh State University, University Square 1, Voronezh 394693, Russia.

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of a random process, despite the possible presence of unstable ordered motion and the fact that the system may be described by a very nontrivial symbolic dynamics [6]. One of these properties is diffusion, which has been approximately established using a variety of approaches [7–21]. A well-studied class of realistic model systems are the kicked-rotor maps on the cylinder

$$\boldsymbol{\Phi}: l_{m+1} = l_m + f(x_m), \quad x_{m+1} = x_m + l_{m+1} \bmod 1 , \tag{1}$$

where l is angular momentum, x is angle ($-0.5 \leqslant x < 0.5$), and the force function $f(x)$ satisfies $f(x+1)=f(x)$ and $f(-x)=-f(x)$. An important property of the maps (1) is their periodicity in both x and l (with period 1). Consider an ensemble $\mathscr{E} = \{(x_0, l_0)\}$ of initial conditions in the chaotic region of unit cell $-0.5 \leqslant x,\ l < 0.5$. We denote by $P_{\mathscr{E}}(l;m)$ the probability distribution (PD) of the angular-momentum values $l = l_m$ after m iterations. For m sufficiently large, one observes an approximately diffusive evolution of $P_{\mathscr{E}}(l;m)$ [9,10]

$$P_{\mathscr{E}}(l;m) \approx \frac{1}{\sqrt{4\pi Dm}} \exp\left(-\frac{l^2}{4Dm}\right) , \tag{2}$$

where D is the diffusion coefficient, which can be calculated directly from

$$D = \lim_{m\to\infty} D_{\mathscr{E}}(m), \quad D_{\mathscr{E}}(m) = \frac{\langle (l_m - l_0)^2 \rangle_{\mathscr{E}}}{2m} \tag{3}$$

with $\langle\,\rangle_{\mathscr{E}}$ denoting average over $\mathscr{E}$.

As implied by Eqs. (2) and (3), both $P_{\mathscr{E}}(l;m)$ and $D_{\mathscr{E}}(m)$ are expected to be essentially independent of the choice of the ensemble $\mathscr{E}$ for sufficiently large m. One is then left with an ambiguity in the choice of "natural" ensembles $\mathscr{E}$ and iteration times m for a systematic calculation of the PDs and D. In addition, for large m the numerical chaotic orbits used in such a calculation differ by 100% from the true orbits due to the roundoff errors caused by the chaotic exponential instability. Thus, in order to establish the existence of chaotic diffusion unambiguously, it is desirable to express the PDs, D, or related quantities in terms of well-defined dynamical entities which can be calculated accurately. These entities were naturally identified in Ref. [12] as the unstable *periodic orbits* (POs) embedded densely in the chaotic region. In a first approach, introduced in Refs. [12,13], chaotic-diffusion rates and PDs for periodic maps were defined on *PO ensembles*. The simplest PD on such an ensemble is given by the exact Eq. (5) below, to be referred to as formula (a). This is a *coarse-grained* PD describing the "diffusion" between unit cells of the periodic map (1). This approach was applied to uniformly hyperbolic systems, the cat and sawtooth maps; exact results were obtained for the diffusion rates [12–14] and the PDs were studied in Refs. [12,15].

A subsequent approach [16–20] leads to an exact PO formula [17,20] for the diffusion coefficient associated with generic ensembles of aperiodic chaotic orbits [see Eq. (6) below]. In Ref. [22], a coarse-grained PD for the diffusion on these ensembles was defined in the case of quasi-one-dimensional periodic chains of coupled billiards and a PO expression for it was derived using general heuristic arguments. In the case of maps (1), this PD and its approximate PO formula (to be referred to as formula (b)) are given by Eqs. (11) and (16) below. Formulas (a) and (b) have been applied to quantum-chaos problems; see, e.g., an application of formula

(a) in Ref. [23] and applications of formula (b) in Refs. [22,24–26]. Refs. [24–26] focus on the $l = 0$ ("return-probability") case of the latter formula.

As far as we are aware, no rigorous proof of formula (b) exists. In addition, a detailed comparison of the results from formulas (a) and (b) with each other and with those from standard methods was apparently never made for a typical, nonuniform hyperbolic system. In the next section, we present a special summary of the PO approaches to chaotic diffusion and we introduce a third definition of a coarse-grained PD (Eq. (14) below). This PD is connected with usual angular-momentum PDs (2) more transparently than the PD defined in Ref. [22]. The approximate formula (b) is then re-derived for the new PD. While this derivation is still nonrigorous, it is more detailed than that in Ref. [22]. We also show the consistency of formula (b) from several points of view. In Section 3, we perform an extensive and very accurate numerical study of formulas (a) and (b) for a completely chaotic system exhibiting a transition from uniform to nonuniform hyperbolicity as a parameter is increased. This is the perturbed Arnol'd cat map on the cylinder. In a recent work [21], PO values of D for this system were calculated in a wide range of perturbation parameters and were found to agree very well with values obtained by standard methods. Here we find that the results from formula (b) in this parameter range fluctuate around the Gaussian distribution (2) and thus agree qualitatively with results from standard methods. The exact formula (a) gives similar results only for sufficiently small parameters, corresponding to almost uniform hyperbolicity. For large parameters, on the other hand, the PD decays monotonically below the Gaussian distribution. However, the deviation between the two distributions seems to gradually disappear as the PO period is increased. These results are expected from general considerations.

2. POs and chaotic diffusion

We start with a summary of the first PO approach to chaotic diffusion, proposed in Ref. [12] (some notation and terms differ from the original ones). A *primitive* PO p of period n for a map (1) is generally defined by initial conditions $(x_0^{(p)}, l_0^{(p)})$ satisfying

$$l_n^{(p)} = l_0^{(p)} + w_p, \quad x_n^{(p)} = x_0^{(p)}, \tag{4}$$

where n is the smallest integer for which (4) holds with integer w_p, the *winding number* of PO p. The general definition (4), allowing for nonzero values of w_p, is based on the periodicity of the maps (1) in both x and l with period 1. Because of this periodicity, (1) can be consistently defined on the unit torus T^2: $-0.5 \leqslant x,\ l < 0.5$ by taking also l modulo 1 and (4) will then reduce to the usual definition of a PO, $l_n^{(p)} = l_0^{(p)}$, $x_n^{(p)} = x_0^{(p)}$. We denote by $\mathscr{U}_n$ the set of all the period-n primitive POs which are distinct when taken modulo T^2. The ensemble $\mathscr{E}_n$ of all the periodic points in $\mathscr{U}_n$ modulo T^2 may be viewed as an invariant "level n" approximation of the chaotic region. A natural choice of $\mathscr{E}$ and m in (2) and (3) is thus $\mathscr{E} = \mathscr{E}_n$ and $m = n$ (in general, one must choose $m \leqslant n$ to avoid periodicity effects [12]). Using (4), this gives the *diffusion rate* on $\mathscr{E}_n$:

$$D(n) \equiv D_{\mathscr{E}_n}(n) = \frac{1}{2nN(n)} \sum_{p \in \mathscr{U}_n} w_p^2 = \frac{1}{2nN(n)} \sum_w N_w(n) w^2 ,$$

where $N(n)$ is the total number of POs in $\mathscr{U}_n$ and $N_w(n)$ is the number of POs in $\mathscr{U}_n$ with $w_p = w$. Similarly, one can associate diffusion rates with subsets of $\mathscr{U}_n$ having well-defined dynamical characteristics [12,13].

The chaotic-diffusion PD on $\mathscr{E}_n$ for $m = n$ is given simply by

$$\mathscr{P}(w; n) \equiv P_{\mathscr{E}_n}(l = w; m = n) = \frac{N_w(n)}{N(n)} . \tag{5}$$

We note that in n iterations of (1), one achieves a perfect "separation" of the subensembles $\mathscr{E}_{n,w}$ of $\mathscr{E}_n$ with given winding number $w : \mathscr{E}_{n,w}$ is fully "transferred" from unit cell T^2 to unit cell $T_w^2 : -0.5 \leqslant x,\ l - w < 0.5$. In this sense, (5) is a *coarse-grained* PD for the "diffusion" of T^2 (represented by $\mathscr{E}_n$) to unit cells T_w^2 (represented by $\mathscr{E}_{n,w}$). The normalization condition $\Sigma_w \mathscr{P}(w; n) = 1$ is trivially satisfied. In addition, $\mathscr{P}(-w; n) = \mathscr{P}(w; n)$, due to the inversion symmetry of (1) $[f(-x) = -f(x)]$, and $\mathscr{P}(w; n) = 0$ for $|w| > w_{\max}(n)$, where $w_{\max}(n)$ is a nondecreasing integer function of n.

A second approach, developed in Refs. [16,17,20], provides an exact PO formula for the diffusion coefficient associated with generic ensembles of aperiodic chaotic orbits in hyperbolic systems:

$$D = -\frac{1}{2} \left. \frac{\partial^2 \zeta^{-1}(\beta, s)/\partial \beta^2}{\partial \zeta^{-1}(\beta, s)/\partial s} \right|_{\beta = s = 0} . \tag{6}$$

Here $\zeta(\beta, s)$ is the Ruelle zeta function [20],

$$\zeta^{-1}(\beta, s) = \prod_p [1 - \exp(\beta w_p - s n_p) |\Lambda_p|^{-1}] , \tag{7}$$

where the product is over all the primitive POs, n_p is the period of PO p, and Λ_p is the associated Lyapunov eigenvalue ($|\Lambda_p| > 1$). A useful PO approximation to D, at given period n, can be easily derived (see, e.g., Ref. [21]) by direct differentiation of the infinite product (7), which is convergent and nonvanishing for $s > 0$ [17,20]:

$$D_{\mathrm{WA}}(n) = \frac{1}{2g(n)} \sum_{p \in \mathscr{U}_n} \frac{|\Lambda_p|^{-1}}{(1 - |\Lambda_p|^{-1})^2} w_p^2 , \tag{8}$$

where

$$g(n) = n \sum_{p \in \mathscr{U}_n} \frac{|\Lambda_p|^{-1}}{1 - |\Lambda_p|^{-1}} . \tag{9}$$

We now consider chaotic-diffusion PDs on generic ensembles. The deterministic time evolution in the cylindrical phase space $\mathbf{z} \equiv (x, l)$ under a completely chaotic map (1) can be expressed by the "microscopic" distribution

$$\rho(\mathbf{z};\mathbf{z}_0;m) = \delta^{(2)}(\mathbf{z} - \boldsymbol{\Phi}^m \mathbf{z}_0)\ , \tag{10}$$

where $\delta^{(2)}(\mathbf{z}) = \delta(x)\delta(l)$ and $\boldsymbol{\Phi}^m \mathbf{z}_0$ is the mth iterate of an initial point $\mathbf{z}_0 = (x_0, l_0)$ under (1). Then, a coarse-grained PD describing the "diffusion" of unit cell T^2 to unit cells $T_w^2 : -0.5 \leqslant x,\ l - w < 0.5$ (w integer) is defined, as in Ref. [22] (Appendix B), by

$$\tilde{P}(w;m) = \int_{T^2} \mathrm{d}\mathbf{z}_0 \int_{T^2} \mathrm{d}\mathbf{z}\ \rho(\mathbf{z} + \mathbf{w};\mathbf{z}_0;m)\ , \tag{11}$$

where $\mathbf{w} \equiv (0, w)$. Unlike the coarse-grained PD (5), which is based on the PO ensemble $\mathscr{E}_n$ in T^2, (11) is based on the "generic" chaotic ensemble given by the entire unit cell T^2. The normalization and inversion-symmetry properties, $\Sigma_w \tilde{P}(w;n) = 1$ and $\tilde{P}(-w;n) = \tilde{P}(w;n)$, are again easily verified.

A different coarse-grained PD, connected with usual angular-momentum PDs (2) more transparently than (11) and also more easily computable than (11), can be defined as follows. Consider a distribution $\hat{\rho}(\mathbf{z};l_0;m)$ which, at time $m = 0$, covers uniformly the horizontal segment $l = l_0$, $-0.5 \leqslant x < 0.5$, i.e., $\hat{\rho}(\mathbf{z};l_0;m=0) = \delta(l - l_0)$. Clearly, at an arbitrary time m,

$$\hat{\rho}(\mathbf{z};l_0;m) = \int_{-0.5}^{0.5} \mathrm{d}x_0\ \rho(\mathbf{z};\mathbf{z}_0;m)\ . \tag{12}$$

The corresponding PD (2) for the angular-momentum displacements $\Delta l = l - l_0$ is given by

$$P_{l_0}(\Delta l;m) = \int_{-0.5}^{0.5} \mathrm{d}x \hat{\rho}(\mathbf{z};l_0;m)\ . \tag{13}$$

It is natural to average (13) over l_0 in the unit cell T^2 ($-0.5 \leqslant l_0 < 0.5$), keeping the displacement Δl fixed. Using (12) and choosing $\Delta l = w$, this gives the coarse-grained PD

$$\bar{P}(w;m) \equiv \int_{-0.5}^{0.5} \mathrm{d}l_0\ P_{l_0}(w;m) = \int_{-0.5}^{0.5} \mathrm{d}x\ \varrho(\mathbf{w}_x;m)\ , \tag{14}$$

where $\mathbf{w}_x \equiv (x, w)$ and

$$\varrho(\mathbf{w}_x;m) \equiv \int_{T^2} \mathrm{d}\mathbf{z}_0\ \rho(\mathbf{z}_0 + \mathbf{w}_x;\mathbf{z}_0;m)\ . \tag{15}$$

Unlike (11), (14) is strictly not normalized, $\sum_w \bar{P}(w;n) < 1$. In practice, however, the normalization holds to high accuracy in most cases (see also below).

An approximate PO formula for (14) can be derived by assuming that $\varrho(\mathbf{w}_x;m)$ is almost constant in the interval $-0.5 \leqslant x < 0.5$ due to quasi-uniformity in one unit cell T_w^2. One can then approximate $\bar{P}(w,m)$ by $\varrho(\mathbf{w}_0;m) = \varrho(\mathbf{w};m)$. For $x = 0$ ($\mathbf{w}_0 = \mathbf{w}$), the integral in (15) can be explicitly expressed in terms of POs using definitions (4) and (10). The final result is

$$\bar{P}(w;n) \approx \varrho(\mathbf{w};n) = n \sum_{n_p r=n,\ w_p r=w} \frac{|\Lambda_p^r|^{-1}}{(1-|\Lambda_p^r|^{-1})^2} , \tag{16}$$

where the quantities n_p, w_p, and Λ_p were defined above and r is an integer (the "repetition index"). Formula (16) is consistent from several points of view. First, the Hannay–Ozorio–de-Almeida uniformity sum rule [20,28] implies that $\lim_{n\to\infty} \sum_w \varrho(\mathbf{w};n)=1$. This is consistent with the approximate normalization $\sum_w \bar{P}(w;n) \approx 1$ which should hold with high accuracy for sufficiently large n. In fact, one expects from (2) that

$$\bar{P}(w;n) \approx E(w;Dn) \equiv \frac{1}{\sqrt{4\pi Dn}} \exp\left(-\frac{w^2}{4Dn}\right) \tag{17}$$

and it is easy to check that the quantity $\Delta(Dn) = 1 - \sum_{w=-\infty}^{\infty} E(w;Dn)$ approaches 0 very rapidly as $Dn \to \infty$ [already $\Delta(1) \approx -10^{-4}$]. Second, the sum rule above implies also that $\lim_{n\to\infty} g(n) = 1$, where $g(n)$ is defined by (9). Thus, for large n, one must have $\sum_w \varrho(\mathbf{w};n)w^2 \approx 2D_{\rm WA}(n)n$, where $D_{\rm WA}(n)$ is given by (8). Since $D_{\rm WA}(n)$ is usually an excellent approximation to the diffusion coefficient D [19,21], this is consistent with the approximate equality $\sum_w \bar{P}(w;n)w^2 \approx 2Dn$ which follows from (17) when $\Delta(Dn) \approx 0$ (large n).

3. Results for the perturbed Arnol'd cat map

In this section, we present extensive and very accurate numerical results for formulas (a) (Eq. (5)) and (b) (Eq. (16)) in the case of the perturbed Arnol'd cat map on the cylinder. This map is given by (1) with

$$f(x) = f_0(x) + \frac{\kappa}{2\pi} \sin(2\pi x) , \tag{18}$$

where $f_0(x)=x$ for $|x| < 0.5$, $f_0(-0.5) \equiv 0$, $f_0(x+1)=f_0(x)$, and κ is a perturbation parameter. This system exhibits a transition from uniform to nonuniform hyperbolicity as κ is "switched on". When defined on the torus T^2, with $f_0(-0.5) \equiv -0.5$, the system is structurally stable for $\kappa < \kappa_c \approx 0.437$ [21] by Anosov theorem [27], i.e., the dynamics for $\kappa < \kappa_c$ is topologically equivalent to that of the unperturbed ($\kappa = 0$) cat map (in particular, the system is completely chaotic like the cat map). Actually, strong numerical evidence indicates that the structural-stability regime extends, at least approximately, beyond κ_c, up to $\kappa \approx 1$ [21]. All the POs with given period in the extended structural-stability regime can be calculated very accurately using the method in Ref. [21]. The relevant dynamics on the cylinder can be easily inferred from that on the torus.

Table 1
$A(w;n)$, $n=9,\ldots,14$, for $\kappa=0.172$

n	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$
9	1.000386	0.995315	1.068721		
10	0.984324	1.015698	1.014111	0.639152	
11	0.994627	1.004482	1.011636	0.885016	
12	0.997686	0.998492	1.031261	0.863434	
13	1.004010	0.993467	1.025119	0.934086	0.313716
14	1.003517	0.993206	1.026763	0.941814	0.519547
$R(w;14)$	0.997218	0.988736	1.043608	0.989682	0.533733

Table 2
$B(w;n)$, $n=9,\ldots,14$, for $\kappa=0.172$

n	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$
9	0.884734	0.863834	0.977635		
10	0.994559	0.996092	1.062360	0.742368	
11	1.002840	0.994307	1.018540	0.987658	
12	0.996496	0.976366	1.039510	0.938571	
13	1.009800	0.984959	1.031250	0.996240	0.398050
14	1.008460	0.985111	1.032900	0.988780	0.654193
$R(w;14)$	0.997218	0.988736	1.043608	0.989682	0.533733

On the basis of the results at the end of the previous section, the natural quantities to be studied will be chosen as follows:

$$A(w;n) \equiv \frac{\mathscr{P}(w;n)}{E[w;\ D_{\mathrm{WA}}(n)n]}\ , \tag{19}$$

$$B(w;n) \equiv \frac{\varrho(\mathbf{w};n)}{E[w;\ D_{\mathrm{WA}}(n)n]}\ , \tag{20}$$

$$R(w;n) \equiv \frac{\bar{P}(w;n)}{E[w;\ D_{\mathrm{WA}}(n)n]}\ , \tag{21}$$

where $\mathscr{P}(w;n)$, $D_{\mathrm{WA}}(n)$, $\bar{P}(w;n)$, $\varrho(\mathbf{w};n)$, and $E(w;Dn)$ are defined by Eqs. (5), (8), (14), (16), and (17), respectively. Because of the inversion symmetry of all these quantities under $w\to -w$, it is sufficient to consider only values of $w\geqslant 0$. Our aim is to examine and compare the closeness of (19)–(21) to 1 for sufficiently large n. We shall consider 11 values of κ in the extended structural-stability regime, $\kappa=0.086k$, $k{=}1,\ldots,11$ ($0.086\leqslant\kappa\leqslant 0.946$). In this regime, the values of $D_{\mathrm{WA}}(n)$ start to converge for $n>8$ and appear to be well converged for $n=14$ [21]. We shall therefore restrict ourselves to periods $9\leqslant n\leqslant 14$. The PD $\bar{P}(w;n)$ in (21) will be calculated from (13) and (14) by standard methods, i.e., by iterating n times a $10^4\times 10^4$ grid of points covering uniformly the torus T^2.

Table 3
$A(w;n)$, $n=9,\ldots,14$, for $\kappa=0.86$

n	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
9	1.178241	0.900216	0.952892	0.672002		
10	1.072393	0.961207	0.983574	0.884015		
11	1.077322	0.977098	0.940772	0.924830	0.366181	
12	1.068613	1.001675	0.918040	0.892517	0.492228	
13	1.059085	1.004719	0.940536	0.889692	0.628971	0.178774
14	1.059269	1.007387	0.946316	0.885802	0.723573	0.231964
$R(w;14)$	1.034109	0.991330	0.980400	1.028447	1.136585	0.682136

Table 4
$B(w;n)$, $n=9,\ldots,14$, for $\kappa=0.86$

n	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
9	1.016970	0.703629	0.920674	1.125140		
10	1.102780	0.912669	0.985949	1.318120		
11	1.082210	0.943077	0.956440	1.249200	0.763700	
12	1.062060	0.949223	0.897202	1.119770	1.534160	
13	1.041070	0.982701	0.957506	1.052660	1.218880	0.672613
14	1.061870	0.968783	0.959416	1.027680	1.358920	0.549543
$R(w;14)$	1.034109	0.991330	0.980400	1.028447	1.136585	0.682136

The results are presented in Tables 1–7. Tables 1 and 2 show results for $A(w;n)$ and $B(w;n)$ in a case of almost uniform hyperbolicity ($\kappa=0.172$). These results appear to agree, at least "qualitatively", with those for $R(w;14)$ (last row), i.e., the values of $A(w;n)$ and $B(w;n)$ fluctuate around 1 like those of $R(w;14)$. This typical behavior of $R(w;14)$ was verified by using many different grids of points in the standard calculation of $R(w;n)$ (see above).

Tables 3 and 4 show results for $A(w;n)$ and $B(w;n)$ in a case of strongly nonuniform hyperbolicity ($\kappa=0.86$). The results for $B(w;n)$ (Table 4) agree again qualitatively with those for $R(w;14)$. In fact, this agreement is observed for all the values of κ considered (compare Table 6 with Table 7). On the other hand, the results for $A(w;14)$ (Table 3) decrease monotonically below 1 as w is increased. As shown in Table 5, this behavior occurs for all values of $\kappa>0.43$ (curiously, $\kappa=0.43$ is close to the Anosov bound $\kappa_c\approx 0.437$ for structural stability [21]). This behavior is usually observed also for $n=12$, 13 and already for $n>10$ in the case of $\kappa=0.86$ (see Table 3). As n is increased, the values of $A(w;n)$ at the "tail" of the distribution ($w=4,5$) always increase, see, e.g., Table 3. This indicates that the monotonic deviation of $A(w;n)$ from the Gaussian distribution may gradually disappear as n is increased.

All these behaviors of $A(w;n)$ and $B(w;n)$ can be easily understood. For sufficiently large κ, the ensemble $\mathscr{E}_n$ of PO points is very nonuniformly distributed in the torus T^2.

Table 5
$A(w; 14)$ for all the 11 values of κ

κ	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
0.086	1.002894	0.992935	1.026411	0.960094	0.411839	
0.172	1.003517	0.993206	1.026763	0.941814	0.519547	
0.258	1.007985	0.990912	1.018720	0.986778	0.494316	
0.344	1.012199	0.991627	1.005531	0.997738	0.650902	
0.43	1.013564	0.994286	1.002189	0.968331	0.713814	
0.516	1.019198	0.994150	0.993153	0.975258	0.747527	
0.602	1.025733	1.001015	0.968402	0.964386	0.846433	0.117387
0.688	1.033537	1.005615	0.959086	0.928352	0.795313	0.423388
0.774	1.046252	1.008874	0.949324	0.895160	0.761977	0.305768
0.86	1.059269	1.007387	0.946316	0.885802	0.723573	0.231964
0.946	1.069435	1.013015	0.935741	0.860602	0.757411	0.210999

Table 6
$B(w; 14)$ for all the 11 values of κ

κ	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
0.086	1.005360	0.988111	1.032250	0.987982	0.496777	
0.172	1.008460	0.985111	1.032900	0.988780	0.654193	
0.258	1.015330	0.981556	1.019930	1.040350	0.696552	
0.344	1.020480	0.981562	1.003670	1.051200	0.949192	
0.43	1.023230	0.980526	1.001680	1.036580	1.065860	
0.516	1.028520	0.979060	0.993053	1.043960	1.124610	
0.602	1.027790	0.986616	0.972304	1.042390	1.234860	0.279826
0.688	1.031140	0.985198	0.975387	1.018380	1.190100	1.086990
0.774	1.050940	0.975193	0.966628	1.007260	1.338820	0.769295
0.86	1.061870	0.968783	0.959416	1.027680	1.358920	0.549543
0.946	1.077580	0.956996	0.976039	0.957068	1.498390	0.532603

Table 7
$R(w; 14)$ for all the 11 values of κ

κ	$w=0$	$w=1$	$w=2$	$w=3$	$w=4$	$w=5$
0.086	1.009226	0.998724	1.035806	0.961328	0.417439	
0.172	0.997218	0.988736	1.043608	0.989682	0.533733	
0.258	0.993201	1.001005	1.015002	1.025753	0.553475	
0.344	1.005112	1.001192	1.017547	1.016591	0.812957	
0.43	0.997703	0.995922	1.001647	1.047210	0.898501	0.061931
0.516	1.018944	0.990968	1.003424	1.036228	0.962447	0.086101
0.602	1.014816	0.991362	0.997245	1.030661	1.078829	0.297583
0.688	1.013913	0.983451	1.003566	1.034428	1.113724	0.665422
0.774	1.024613	0.992281	0.998433	1.005856	1.085715	0.573604
0.86	1.034109	0.991330	0.980400	1.028447	1.136585	0.682136
0.946	1.044070	0.979206	0.974599	1.007814	1.275581	0.677513

The highest concentration of PO points occurs near $x = 0$, where the local instability, measured by the Lyapunov eigenvalues Λ_p, is the largest. On the other hand, PO points near $|x| = 0.5$ are most sparse since the local instability is the smallest here. The stability factors $|\Lambda_p^r|^{-1}(1 - |\Lambda_p^r|^{-1})^{-2}$ in (16), which are absent in (5), precisely compensate for this nonuniformity and lead to an almost uniform "weighted" density of PO points approximating the uniform density of generic ensembles of aperiodic chaotic orbits [20]. Now, the force function (18) assumes its smallest (largest) absolute values near $x = 0$ ($|x| = 0.5$). Thus, the relative size of subensembles $\mathscr{E}_{n,w}$ with small (large) $|w|$ will be larger (smaller) than that of the corresponding subensembles in generic uniform ensembles. This explains the monotonic decay of $A(w; n)$ below the Gaussian distribution. As n is increased, this deviation between the two distributions is expected to decrease since the ensemble $\mathscr{E}_n$ becomes more uniformly distributed in T^2.

In conclusion, the approximate PO formula (16) for PDs on generic ensembles gives results agreeing qualitatively with those from standard methods. It is still an open problem how to improve this formula by removing, at least partially, the approximations made in going from Eq. (14) to (16). On the other hand, formula (5) for PDs on PO ensembles is exact by definition. For nonuniform hyperbolicity, PO ensembles are nonuniformly distributed in phase space. The results from formula (5) may then deviate from an expected "generic" Gaussian distribution. In principle, however, PO ensembles are completely legitimate and also dynamically natural and well defined. Thus, the exact formula (5) seems to provide the simplest and most systematic approach to chaotic-diffusion PDs at present.

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