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Chaotic diffusion on periodic orbits and uniformity

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Abstract

Chaotic diffusion on periodic orbits (POs) is studied for the perturbed Arnol'd cat map, exhibiting a transition from uniform to nonuniform hyperbolicity as the perturbation parameter is increased. The results for the diffusion coefficient from PO formulas agree very well with those obtained by standard methods. Using the original PO formula involving a uniform (nonweighted) average over the POs, reasonably accurate results are obtained for sufficiently small parameters corresponding also to cases of a considerably nonuniform hyperbolicity. This is due to *uniformity sum rules* satisfied by the PO Lyapunov eigenvalues at *fixed winding number*.

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An important manifestation of the randomness of deterministic chaos is diffusion, whose existence in Hamiltonian systems has been approximately established using a variety of approaches [1–12]. A systematic approach, first proposed in Refs. [6,7] and developed further in subsequent works [8–12], is based on the hierarchy of the unstable periodic orbits (POs) embedded in the chaotic region. Let us summarize the main ideas in Ref. [6] by considering, for definiteness, the kicked-rotor maps on the cylinder

$$l_{m+1} = l_m + f(x_m), \quad x_{m+1} = x_m + l_{m+1} \mod 1 , \tag{1}$$

where l is angular momentum, x is angle, and the force function $f(x)$ satisfies $f(x+1) = f(x)$ and $f(-x) = -f(x)$. The diffusion coefficient for (1) is formally

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defined by

$$D = \lim_{m\to\infty} D_{\mathscr{E}}(m), \quad D_{\mathscr{E}}(m) = \frac{\langle (l_m - l_0)^2 \rangle_{\mathscr{E}}}{2m} , \tag{2}$$

where $\langle \ \rangle_{\mathscr{E}}$ denotes average over an ensemble $\mathscr{E} = \{(x_0, l_0)\}$ of initial conditions in a chaotic region. The ambiguity in the choice of $\mathscr{E}$ and m in a finite-time approximation $D_{\mathscr{E}}(m)$ is systematically fixed [6] by choosing $\mathscr{E}$ as the ensemble $\mathscr{U}_n$ of all the primitive POs of period n in the chaotic region and $m = n$. Since the map (1) is periodic in (x, l) with period 1, a PO $p \in \mathscr{U}_n$ is generally defined by initial conditions $(x_0^{(p)}, l_0^{(p)})$, $-0.5 \leqslant x_0^{(p)}, l_0^{(p)} < 0.5$, satisfying

$$l_n^{(p)} = l_0^{(p)} + w_p, \quad x_n^{(p)} = x_0^{(p)} , \tag{3}$$

where n is the smallest integer for which (3) holds with integer w_p. The quantity w_p is the *winding number* of PO p. The ensemble $\mathscr{U}_n$ may be viewed as an invariant "level n" approximation of the chaotic region. Because of (3), the diffusion rate $D(n) = D_{\mathscr{E}=\mathscr{U}_n}(m = n)$ is given by

$$D(n) = \frac{1}{2nN(n)} \sum_{p\in\mathscr{U}_n} w_p^2 = \frac{1}{2nN(n)} \sum_{w} N_w(n) w^2 , \tag{4}$$

where $N(n)$ is the total number of POs in $\mathscr{U}_n$ and $N_w(n)$ is the number of POs in $\mathscr{U}_n$ with $w_p = w$. Similarly, one can associate diffusion rates with subensembles of $\mathscr{U}_n$ having well-defined dynamical characteristics [6,7]. For uniformly hyperbolic systems, (4) is expected to approximate well the diffusion coefficient (2) associated with generic ensembles of aperiodic chaotic orbits. In fact, (4) gives the exact value of D for the cat maps and approximates very well D for the sawtooth maps [6].

The diffusion coefficient for generic chaotic ensembles in hyperbolic systems is given by the exact PO formula [9,13,14]

$$D = -\left. \frac{1}{2} \frac{\partial^2 \zeta^{-1}(\beta, s)/\partial\beta^2}{\partial\zeta^{-1}(\beta, s)/\partial s} \right|_{\beta=s=0} . \tag{5}$$

Here $\zeta(\beta, s)$ is the Ruelle zeta function [14],

$$\zeta^{-1}(\beta, s) = \prod_p \left[1 - \exp(\beta w_p - s n_p)|\Lambda_p|^{-1}\right] , \tag{6}$$

where the product is over all the primitive POs, n_p is the period of PO p, and Λ_p is the associated Lyapunov eigenvalue ($|\Lambda_p| > 1$). One can express (6) as a power series in $\exp(-s)$:

$$\zeta^{-1}(\beta, s) = 1 + \sum_{n=1}^{\infty} c_n(\beta) \exp(-sn) , \tag{7}$$

where the $n > 1$ terms are known as "curvatures" [13]. The convergence of (7) is generally better than that of the infinite product (6). The application of (5) and related

PO formulas [8,13] to a realistic nonuniformly hyperbolic system, the periodic Lorentz gas, gives results [11] that are only within 8% of the values of D obtained by standard methods.

In this work, chaotic diffusion on POs is studied for a nontrivial Hamiltonian system exhibiting a transition from uniform to nonuniform hyperbolicity as a parameter is varied. This is the perturbed Arnol'd cat map on the cylinder, defined by (1) with

$$f(x) = f_0(x) + \frac{\kappa}{2\pi}\sin(2\pi x)\,, \tag{8}$$

where $f_0(x)=x$ for $|x| < 0.5$, $f_0(-0.5) \equiv 0$, $f_0(x+1)=f_0(x)$, and κ is a perturbation parameter. This system, with the definition $f_0(-0.5) \equiv -0.5$, is usually considered on a torus, $-0.5 \leqslant x, l < 0.5$. The unperturbed ($\kappa = 0$) map is uniformly hyperbolic and completely chaotic but already features a very nontrivial symbolic dynamics with non-explicit pruning rules given by an infinite set of inequalities [15]. As a result, all the curvature terms in zeta-function expansions are nonvanishing. Anosov theorem [16] states that the dynamics on the torus for sufficiently small κ, $\kappa < \kappa_c$, is topologically equivalent to that of the unperturbed system. This expresses the well-known structural stability of cat maps. Actually, strong numerical evidence [17] indicates that the structural-stability regime extends, at least approximately, much beyond $\kappa_c \approx 0.437$, up to $\kappa \approx 1$; the system remains completely chaotic up to $\kappa \approx 1.5$.

The POs on the torus in the extended structural-stability regime are calculated very accurately as follows. First, the POs $\mathcal{O}_0$ of the unperturbed map are determined exactly using the techniques in Ref. [18]. The topologically-equivalent POs $\mathcal{O}_\kappa$ in the perturbed case are then computed as $\mathcal{O}_\kappa = H_\kappa \mathcal{O}_0$, where the map H_κ is constructed iteratively from a nonlinear functional equation satisfied by it, starting from the solution $H_\kappa^{(0)}$ of a "homological equation" (see details in the proof of Anosov theorem [16]). In this way, we have calculated all the perturbed POs $\mathcal{O}_\kappa$ with periods $n \leqslant 14$ for $\kappa \leqslant 0.946$ with an accuracy of at least 10^{-10}; this accuracy was checked by direct iteration of the map. The relevant POs on the cylinder are easily inferred from those on the torus. Since essentially no bifurcations can take place in the extended structural-stability regime ($\kappa \leqslant 0.946$), the variation of the diffusion coefficient D with κ is totally due to the change of the characteristics (w_p, Λ_p) of a constant number of POs. Thus, the case studied here is basically different from that considered in Ref. [10], i.e., standard maps in a strong-chaos limit. In the latter case, bifurcations of small-period ($n = 1$ and 2) POs are the main cause for the relevant variation of D with the parameter.

We calculate PO approximations to D for $n \leqslant 14$ and $\kappa = 0.086k$, $k = 1, \ldots, 11$ ($0.086 \leqslant \kappa \leqslant 0.946$), using three formulas: (a) The curvature-expansion (CE) approximation, $D_{\mathrm{CE}}(n)$, obtained by using in (5) expansion (7) truncated after the first n terms. (b) The "weighted-average" (WA) approximation

$$D_{\mathrm{WA}}(n) = \frac{1}{2g(n)} \sum_{p\in\mathcal{U}_n} \frac{|\Lambda_p|^{-1}}{(1-|\Lambda_p|^{-1})^2}\; w_p^2, \qquad g(n) \equiv n \sum_{p\in\mathcal{U}_n} (|\Lambda_p| - 1)^{-1} \tag{9}$$

which can be easily inferred from (5) and (6) after simple algebra using the inversion symmetry of the map (1) [$f(-x) = -f(x)$]. For sufficiently large n, (9) is just the average of $w_p^2/(2n)$ ($p \in \mathcal{U}_n$) weighted by the stability factor $|\Lambda_p|^{-1}$. Such

Table 1
$D_{CE}(14)$, $D_{WA}(14)$, $D(14)$, D_S, $\lambda_{min}(14)$, and $\lambda_{max}(14)$ for all the 11 values of κ

κ	$D_{CE}(14)$	$D_{WA}(14)$	$D(14)$	D_S	$\lambda_{min}(14)$	$\lambda_{max}(14)$
0.086	0.04350	0.04377	0.04356	0.04388	0.9358	0.9927
0.172	0.04551	0.04618	0.04579	0.04622	0.9092	1.0221
0.258	0.04824	0.04861	0.04815	0.04865	0.8827	1.0508
0.344	0.05070	0.05109	0.05046	0.05116	0.8563	1.0787
0.43	0.05338	0.05378	0.05273	0.05380	0.8300	1.1060
0.516	0.05589	0.05642	0.05506	0.05656	0.8039	1.1327
0.602	0.05841	0.05938	0.05736	0.05942	0.7779	1.1588
0.688	0.06220	0.06228	0.05921	0.06240	0.7520	1.1844
0.774	0.06531	0.06548	0.06102	0.06551	0.7262	1.2094
0.86	0.06838	0.06848	0.06297	0.06873	0.7006	1.2339
0.946	0.07188	0.07180	0.06505	0.07200	0.6751	1.2580

approximations to D have been used in previous works [8,11]. (c) The nonweighted-average formula (4). The PO results for D are compared with standard ones obtained from (2) by choosing $\mathscr{E}$ as the entire unit torus. For this invariant ensemble, one has the exact expansion [2]

$$D_{\mathscr{E}}(m) = \frac{1}{2}C_0 + \sum_{j=1}^{m-1}\left(1 - \frac{j}{m}\right)C_j, \quad m > 1 , \tag{10}$$

where $C_j = \langle f(x_0)f(x_j)\rangle_{\mathscr{E}}$ are the force–force correlations for (8). These correlations are calculated very accurately for $j \leqslant 30$ by a sophisticated integration of $f(x_0)f(x_j)$ over the unit torus. In general, $D_{\mathscr{E}}(m)$ in (10) converges rapidly to D due to the fast decay of C_j. For example, $D_{\mathscr{E}}(20)$ differs from both $D_{\mathscr{E}}(10)$ and $D_{\mathscr{E}}(30)$ by no more than 0.05% for all the values of κ considered. In what follows, $D_{\mathscr{E}}(20)$ will serve as our "standard" value D_S for D.

The quantities $D_{CE}(14)$, $D_{WA}(14)$, $D(14)$, and D_S are listed in Table 1 for all the 11 values of κ considered. The very good agreement between $D_{WA}(14)$ and D_S is quite evident. This agreement is generally better than that between $D_{CE}(14)$ and D_S. The relative difference between $D_{WA}(14)$ and D_S ranges from 0.04% to 0.4% while that between $D_{CE}(14)$ and D_S ranges from 0.16% to 1.7%. Thus, the PO results are certainly more accurate than those in the case of the Lorentz gas [11].

Table 1 also shows that the relative difference between the values of $D(14)$ and D_S for $\kappa \leqslant 0.43$ is not larger than 2% despite the fact that the hyperbolicity for $\kappa \leqslant 0.43$ can be considerably nonuniform. This can be seen from the last two columns in Table 1, which give the minimal and the maximal value of the PO Lyapunov exponent $\lambda_p = \ln(|\Lambda_p|)/n$ for $n = 14$ (for $\kappa = 0$, $\lambda_{min} = \lambda_{max} \approx 0.9624$). We see, for example, that for $\kappa = 0.43$ the maximal value of $|\Lambda_p(14)|$ is about 50 times larger than its minimal value.

To understand this, let us first write (9) as follows:

$$D_{WA}(n) = \frac{1}{2nN(n)}\sum_w N_w(n)S_w(n)w^2 , \tag{11}$$

where $N(n)$ and $N_w(n)$ are defined as in (4) and

$$S_w(n) = \frac{nN(n)}{g(n)N_w(n)} \sum_{p\in\mathscr{U}_n,\ w_p=w} \frac{|\Lambda_p|^{-1}}{(1-|\Lambda_p|^{-1})^2} \; . \tag{12}$$

Next, let us assume the *uniformity sum rules* at *fixed winding number w*,

$$\lim_{n\to\infty} S_w(n) = 1 \; . \tag{13}$$

We found much numerical evidence for the validity of (13) in our system [17]. In general, the origin of (13) can be understood as follows. For $n \gg 1$, one has the approximate relation (see, e.g., Appendix B in Ref. [19]):

$$\frac{1}{\sqrt{4\pi nD}} \exp\left(-\frac{w^2}{4Dn}\right) \approx n \sum_{n_p r=n,\ w_p r=w} \frac{|\Lambda_p^r|^{-1}}{(1-|\Lambda_p^r|^{-1})^2} \; , \tag{14}$$

where r (an integer) is the repetition index. The left-hand side of (14) gives the probability distribution for a generic chaotic ensemble to diffuse a “distance” $|w|$ in “time” n. Now, as $n \to \infty$, there should be no essential difference between such an ensemble and the PO ensemble $\mathscr{U}_n$. The probability distribution above is then expected to be approximately equal to $N_w(n)/N(n)$ provided $|w|$ is not too close to its maximal value [6]. We use this in (14), keeping only the dominant terms ($r=1$) on the right-hand side. Recalling also the definition (12) and the well-known rule $\lim_{n\to\infty} g(n)=1$ [14,20], Rel. (13) is obtained.

Now, the effect of a nonuniform hyperbolicity in (11) is completely captured by quantities (12). This effect manifest itself only if $S_w(n)$ is not well converged to 1. For sufficiently large n, this occurs only around the “tail” of distribution (14) ($|w|$ relatively close to its maximal value), due to a small number $N_w(n)$ of POs with $w_p = w$. Precisely because of the last fact, however, the “remainder” $N_w(n)[S_w(n) - 1]$ in (11) may turn out to be negligible. Then, formula (9) will reduce essentially to the nonweighted-average formula (4) despite of a considerably nonuniform hyperbolicity. This is, in fact, the case for $\kappa \leqslant 0.43$.

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