Quantum Walks: First Detected Passage Time

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Outline

• Classic random walk
• Goal
• Quantum walks
• The measurement process
• Z transform
• Results:
  • Rings
  • Infinite lattices
Classic random walk
Classic first passage time

• A fundamental question about a classical random walker: How long does it take it to reach a certain region for the first time?

• Renewal equation:

\[ P(\vec{r}, t) = \delta_{r,0}\delta_{t,0} + \sum_{t'=0}^{t} F(\vec{r}, t') P(0, t - t'), \]

relates between spatial distribution \( P(\vec{r}, t) \) and first passage time distribution \( F(\vec{r}, t) \).
Classic first passage time

• For a 1D Brownian motion, the first passage time distribution is:

\[ F(x_0, t) = \frac{x_0}{\sqrt{4\pi Dt}} e^{-\frac{x_0^2}{4Dt}}. \]

• It has a fat tail of \( t^{-3/2} \)

\[
\int_0^\infty F(x_0, t) dt = 1
\]

\[
\int_0^\infty tF(x_0, t) dt \to \infty
\]

\[ x_0 = 6, D = 0.5 \]
Goal – quantum FPT

• We want to discuss the problem of first passage time of a quantum particle.

• Problems:
  • Position has no binary meaning, so “passage” is ambiguous.
  • There is no first passage time operator.
Quantum walks

$|\langle x | \psi(\gamma t = 50) \rangle|^2$
Quantum walk

- There are two types of quantum walks:
  - Quantum coin
  - Continuous propagation by the Hamiltonian

- For an infinite lattice, with initial condition localized on the origin and tight-binding Hamiltonian $H = -\gamma \sum_{l=-\infty}^{\infty} (|l\rangle\langle l+1| + |l+1\rangle\langle l|)$, the propagation, according the Schrodinger equation is ($\hbar = 1$):
  \[
  \langle x|\psi(t)\rangle = (-i)^x J_x(2\gamma t).
  \]
Quantum first detected passage time

• We discuss a system described by a time independent Hamiltonian $\hat{H}$ and a propagator $U_t = \exp(-iHt)$.

• A region of the lattice is observed stroboscopically with time interval $\tau$. For simplicity, the detected region will contain a single site, $|0\rangle$.

• The measurement has two possible outcomes: the particle is either detected or it isn’t. We will get a sequence of no, no, no, no, ..., yes.
Quantum first detected passage time

• The first detected passage time: What is the probability that the first successful measurement of the particle is the $n$-th measurement?

• Spoiler: We will get a quantum renewal equation for the amplitudes on the detected site before the measurements:

$$\phi_n = \langle 0 | U_{n\tau} | \psi(0) \rangle - \sum_{k=1}^{n-1} \langle 0 | U_{(n-k)\tau} | 0 \rangle \phi_k$$

$$F_n = |\phi_n|^2$$
The measurement process
The first measurement

• The wave function propagates freely for time $\tau$

$$|\psi(\tau^-)\rangle = U_\tau |\psi(0)\rangle$$

• Then the site $|0\rangle$ is measured, and the particle is detected there with probability

$$P_1 = |\langle 0|\psi(\tau^-)\rangle|^2$$

• If the particle is detected, the experiment is finished. Otherwise, the wave collapses in $|0\rangle$ and the rest of renormalized:

$$|\psi(\tau^+)\rangle = (1 - P_1)^{-1/2} (\hat{1} - |0\rangle\langle 0|) |\psi(\tau^-)\rangle$$
Later measurements

- The process is continued, the waveform propagates for time \( \tau \), collapses in \( |0\rangle \) and renormalized.

\[
|\psi(n\tau^-)\rangle = U_\tau |\psi((n - 1)\tau^+)\rangle
\]

\[
P_n = |\langle 0 |\psi(n\tau^-)\rangle|^2
\]

\[
|\psi(n\tau^+)\rangle = (1 - P_n)^{-1/2} (\hat{1} - |0\rangle\langle 0|) |\psi(n\tau^-)\rangle
\]
First detection amplitudes

• We can factor the previous renormalizations by:

\[
\phi_n = \left[ \prod_{k=1}^{n-1} (1 - P_k)^{+1/2} \right] \langle 0 | \psi(n\tau^-) \rangle
\]

• Together with

\[
F_n = \prod_{k=1}^{n-1} (1 - P_k)P_n
\]

we get

\[
F_n = |\phi_n|^2.
\]
Calculating $\phi_n$

• To sum up, the first detection wave function propagates once for time $\tau$, and ever since that it collapses in $|0\rangle$ and propagate again, $n - 1$ times.

$$\phi_n = \langle 0|[U_\tau (1 - |0\rangle\langle 0|)]^{n-1}U_\tau |\psi(0)\rangle$$

• Equivalently, we get the following quantum renewal equation:

$$\phi_n = \langle 0|U_{n\tau}|\psi(0)\rangle - \sum_{k=1}^{n-1} \langle 0|U_{(n-k)\tau}|0\rangle \phi_k$$
Z transform
The Z transform

- We now use the following Z transform:

\[ \hat{\phi}(z) = \sum_{n=1}^{\infty} \phi_n z^n, \quad (n \to z) \]

to see that

\[ \hat{\phi}(z) = \frac{\langle 0|\tilde{U}(z)|\psi(0)\rangle}{1 + \langle 0|\tilde{U}(z)|0\rangle} \]

where \( \tilde{U}(z) \) is

\[ \tilde{U}(z) = \frac{1}{z^{-1}\exp(iH\tau) - 1} \]
Quantum Zeno effect

• In the $\tau \to 0$ limit the $\hat{\phi}(z)$ equation reduces to:
  \[ \hat{\phi}(z) = \langle 0|\psi(0)\rangle z + O(\gamma\tau) \]

• This is linear by $z$, so the particle can be detected only in the first attempt, and its probability is determined by the projection of the initial wave function over the detected site.
  \[ \phi_1 = \langle 0|\psi(0)\rangle, \phi_{n>1} = 0 \]

• This is called the quantum Zeno effect.
Energy spectrum

• In the energy spectrum, $\hat{U}(z)$ is diagonal.

• We now treat tight-binding Hamiltonian for a ring with $L$ sites:

$$H = -\gamma \sum_{l=0}^{L-1} (|l\rangle\langle l+1| + |l+1\rangle\langle l|), \quad |L\rangle \equiv |0\rangle$$

• For a localized initial condition $|\psi(0)\rangle = |x_0\rangle$:

$$\hat{\phi}(z) = \frac{1}{L} \sum_k e^{-2i\pi\frac{kx_0}{L}} \left[ z^{-1} \exp \left( 2i\gamma \tau \cos \frac{2\pi k}{L} \right) - 1 \right]^{-1}$$

$$= \frac{1}{1 + \frac{1}{L} \sum_k \left[ z^{-1} \exp \left( 2i\gamma \tau \cos \frac{2\pi k}{L} \right) - 1 \right]^{-1}}$$
Recovering properties from $\hat{\phi}(z)$

$$\phi_n = \frac{1}{2\pi i} \int z^{-n-1} \hat{\phi}(z) dz$$

$$1 - S_\infty = \sum_{n=1}^{\infty} |\phi_n|^2 = \frac{1}{2\pi} \int_{0}^{2\pi} |\hat{\phi}(e^{i\theta})|^2 d\theta$$

$$\langle n \rangle = \sum_{n=1}^{\infty} nF_n = \frac{1}{2\pi} \int_{0}^{2\pi} \left[\hat{\phi}(e^{i\theta})\right]^{*} \left(-i \frac{\partial}{\partial \theta}\right) \hat{\phi}(e^{i\theta}) d\theta$$
Interim summary

• We described a simple quantum system with projective measurements.

• We showed that the first detection probability is related to the not renormalized wave function projection on the detected site.

• We found the $Z$ transform of the not-renormalized waveform for a tight-binding ring with $L$ sites.

• We can inverse the $Z$ transform in order to get the first detection properties.
Results
Rings: $|0\rangle \rightarrow |0\rangle$ transition

1) A particle which started on the detected site will be detected with probability 1. It does not depend on energy spectrum.

2) The average number of detection attempts is:

$$\langle n \rangle = \begin{cases} 
(L + 2)/2, & L \text{ is even} \\
(L + 1)/2, & L \text{ is odd} 
\end{cases}$$

3) There are exceptional $\tau$ values, where the average is lower.

$$\Delta E\tau = 2\pi n$$
\[ \langle n \rangle_N \]

\[ \gamma \tau \]
Benzene ring

• Total detection probabilities \((1 - S_\infty)\) for various initial conditions:

<table>
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<th>(x_0)</th>
<th>(0 &lt; \gamma \tau &lt; 2\pi^*)</th>
<th>0</th>
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<th>(\frac{2}{3}\pi)</th>
<th>(\pi)</th>
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</tr>
</tbody>
</table>

• Share exceptional \(\tau\) values with \(\langle n \rangle\) where \(x_0 = 0\).
Benzene ring, $|3\rangle \rightarrow |0\rangle$ transition

- Diverges where $S_\infty \neq 0$.

- Has analytic form:
  \[
  \langle n \rangle = \frac{27 + 23 \cos(\gamma \tau) + 24 \cos(2\gamma \tau) + 9 \cos(3\gamma \tau) - 2\cos(4\gamma \tau)}{9 \sin^2(2\gamma \tau)}
  \]

- Singular point where the equation is not valid is $\gamma \tau = \frac{2}{3} \pi$, although $S_\infty = 0$. 
Benzene ring, $|3\rangle \rightarrow |0\rangle$ transition
Infinite lattice

• For infinite lattice, $L \to \infty$, (initial waveform is $|\psi(0)\rangle = |0\rangle$) we get:

$$\hat{\phi}(z) = 1 - \frac{1}{\sum_{n=0}^{\infty} z^n J_0(2\gamma \tau n)}$$

• Small $n$s can be calculated by expanding $\hat{\phi}(z)$ to Taylor series of $z$.

• Large $n$s can be calculated asymptotically.
Infinite lattice, large $n$s

• After a long calculation, one gets:

$$F_n \approx \frac{4\gamma\tau}{\pi n^3} \cos^2 \left(2\gamma\tau n + \frac{\pi}{4}\right)$$

for all $\gamma\tau$, except where $\gamma\tau = \frac{\pi}{2} k$, where we get:

$$F_n \bigg|_{\gamma\tau = \frac{\pi}{2} k/2} \approx \frac{k}{4n^3} = \frac{1}{4} \lim_{\gamma\tau \to \pi k/2} F_n$$
\[ \gamma \tau = 0.8 \]
\( \gamma \tau = \pi / 3 \)
$1 - S_\infty$
Summary

• The first detection event statistics of a quantum system with periodic projective measurements can be successfully treated with our quantum renewal equation.

• Our formalism allows us to discover several quantum behaviors:
  • Rings have a strong dependence on the initial condition, and singularities at specific observation periods.
  • Infinite lattices has a power law decay of $n^{-3}$ with oscillations.
Thank you for listening!