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Anomalous Diffusion in the Strong Scattering Limit: A Lévy Walk Approach

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Abstract. The continuous time random walk (CTRW) is a powerful stochastic theory developed and used to analyze regular and anomalous diffusion. In particular this framework has been applied to sublinear, dispersive, transport and to enhanced Lévy walks. In its earlier version the CTRW does not include the velocities of the walker explicitly, and therefore it is not suited to analyze situations with randomly distributed velocities. Experiments and theory have recently considered systems which exhibit anomalous diffusion and are characterized by an inherent distribution of velocities. Here we develop a modified CTRW formalism, based on a velocity picture in the strong scattering limit, with emphasis on the Lévy walk limit. We consider a particle which randomly collides with unspecified objects changing randomly its velocity. In the time intervals between collision events the particle moves freely. Two probability density functions (PDF) describe such a process: (a) $q(\tau)$, the PDF of times between collision events, and (b) $F(v)$, the PDF of velocities of the particle. In this renewal process both the velocity of the random walker and the time intervals between collision events are independent, identically distributed, random variables. When either $q(\tau)$ or $F(v)$ are long-tailed the diffusion may become non-Gaussian. The probability density to find the random walker at $r$ at time $t$, $\rho(r,t)$, is found in Fourier-Laplace space. We discuss the role of initial conditions especially on the way $P(v,t)$, the probability density that the particle has a velocity $v$ at time $t$, decays to its equilibrium. The phase diagram of the regimes of enhanced, sublinear and normal types of diffusion is presented. We discuss the differences and similarities between the Lévy walk collision process considered here and the CTRW for jump processes.

1 Introduction

Anomalous diffusion [1-5] is a well established phenomenon, found in a broad range of fields. It is characterized by

$$\langle r^2 \rangle \sim t^\beta$$

(1)

with $\beta \neq 1$. Various mechanisms are known which lead to enhanced diffusion ($\beta > 1$), or to subdiffusion, also called dispersive or slow diffusion ($\beta < 1$). For these cases the goal is to find the probability density $\rho(r,t)$ to be at $r$ at time $t$, from which transport moments can be obtained. Stochastic frameworks which describe phenomenologically such behaviors include fractal Brownian motion [6], fractional calculus [7-12], generalized diffusion equation [13-17] and a generalized Langevin equation approach [18, 19]. Two other approaches, relevant especially
here, are: (a) the continuous time random walk (CTRW), [20, 21] and (b) a velocity approach whose details are given below [2, 22-28]. Our main aim in this work is to generalize the velocity approach and show its relation to the CTRW.

All these stochastic theories are used to describe processes for which the conditions that ensure the validity of the central limit theorem (CLT) are not satisfied. Lévy and Khintchine [29] introduced a generalization of the CLT to the case where \( x_i \) in the sum \( X_N = \sum_{i=1}^{N} x_i \) are independent, identically distributed, random variables with a long-tailed distribution so that the existence of the first two moments is not necessarily assumed. The corresponding random walk, \( X_N \), is called a Lévy flight. For such random walks the mean squared displacement \( \langle X_N^2 \rangle \) diverges for all \( N > 0 \) and, hence, a Lévy flight cannot represent an anomalous diffusion of the type in Eq. (1).

A way to overcome the divergence in the mean squared displacement is to introduce a velocity into the random walk scheme, the result being that the divergence of the mean square displacement found for Lévy flights is replaced by enhanced diffusion (see e.g. [25]). Such random walks are called Lévy walks [23] and can be defined within the context of the CTRW. Within the CTRW framework the pausing times between successive steps, as well as the lengths of the steps, are random variables. Schemes in which the pausing times and step lengths are either decoupled or coupled [1, 30] have been investigated thoroughly [20, 21]. For both cases the jumping events are instantaneous so that the random walker pauses at some location for a finite (random) time and then performs a jump, with a vanishing jumping time, to another location. Within the CTRW the stochastic evolution is described in terms of \( \psi(r, t) \), the probability density of making a jump of length \( r \) in the time interval between \( t \) and \( t + dt \). Klafter, Blumen and Shlesinger [30] introduced a coupled space-time distribution

\[
\psi_{ct}(r, t) = Cr^{-\mu_{ct}} \delta(r - t^{\nu_{ct}})
\]

where, through the Dirac \( \delta \) function, \( r \) and \( t \) are coupled. The idea behind this coupling is that the longer is the step, the more time it takes to be performed. For this coupling one can find normal, enhanced or sublinear types of diffusion, depending on the choice of exponents \( \mu_{ct} \) and \( \nu_{ct} \) (the subscript \( ct \) stands for CTRW). Recently, in [31] such a coupled CTRW was used to analyze statistical properties of chaotic trajectories generated by a deterministic non-linear map.

Although the jump process described by the CTRW with space-time coupling, Eq. (2), is a powerful tool describing anomalous diffusion, it considers explicitly the positions of the random walkers and not their velocities. Therefore it is less suited to describe systems where the velocity of the random walkers is an important stochastic ingredient of the transport process. However, recently much attention has been drawn to systems which exhibit anomalous type of diffusion, and for which the velocities of the walker are randomly distributed according to a PDF \( F(v) \) [32-36].

Approaches which include a constant velocity were developed [2, 22-25], where a particle moves with a constant speed for some time interval and then a new direction of motion is chosen (the kinetic energy of the particle is a constant of
motion). In one-dimension such a model is a two state model with the velocity having only the values ±v. When the times of free motion with a constant velocity are independent random variables whose distribution decays algebraically, then, under certain conditions, the diffusion is enhanced and displays a Lévy walk.

Here we consider the more general case for which the magnitude of the velocity is not necessarily constant, but may rather change due to collision events. We investigate a d-dimensional stochastic strong collision model, according to which a particle moves freely between turning points (collision events), and at each turning point the velocity of the particle (both direction and magnitude) is randomized. The process is renewed after each collision. The time intervals between collision events are described by a probability density function (PDF) \( q(\tau) \) and the velocities of the particle are described by a PDF \( F(v) \). Both the times and the velocities are assumed to be independent, identically distributed, random variables. We call such a process a Lévy walk collision process (LWCP).

An example for such a process is the anomalous Knudsen diffusion proposed by Levitz [37], where the velocities of the gas particles are Maxwell distributed and scatterers are randomly distributed on a fractal structure in such a way that the PDF of the times between collision events decays algebraically. Other examples can be found in non-linear chaotic dynamics and in turbulent systems [32, 35, 36] and also in the statistics of single ion trajectories in optical lattices which exhibit Lévy walks [27, 34].

In figures (1) and (2) two different one-dimensional realizations of the LWCP are shown for the case where the mean time between collisions diverges,

\[
q(\tau) \sim \tau^{-3/2}.
\]  

(3)

As can be seen the velocity is not a continuous function of time due to the strong collision events. Figure (1) shows a realization of the LWCP for the case where \( F(v) \) is a Gaussian and this should be compared with the two state velocity model \( \{i.e. F(v) = 0.5[\delta(v - 1) + \delta(v + 1)]\} \) shown in figure (2) and considered previously in [25]. Characteristic of the stochastic process shown in figures (1) and (2) are long time intervals in which no collision events take place. In appendix A we provide a short algorithm which generates time intervals whose PDF decays algebraically with time, Eq. (3) being an example.

When the collision times are Poisson distributed and the velocity PDF decays fast enough the LWCP reduces to the well known strong collision model (the Drude model) widely used in the context of plasma and condensed matter physics [38, 39].

If the time between collision events is a constant \( \tau_0 \), the number of collision events in the interval \( (0, t) \) is a non random variable \( N \). For this case the LWCP reduces to a Lévy flight with a diverging mean squared displacement, provided that the first or the second moments of the velocity PDF diverge. This case has been investigated recently by Zanette and Alemany [40] (and see also [41]).
Fig. 1. Velocity vs time (dimensionless units) for a one-dimensional Lévy walk collision process. We generate such a process by generating two random numbers per collision. The first is the velocity of the particle, described by the PDF $F(v)$, and the second is the time of free motion described with the PDF $q(\tau)$ Eq. (3). Here the particle has encountered 200 collisions and the velocity PDF $F(v)$ is a Gaussian with a variance that equals unity. Notice the long time intervals in which no collision takes place, this is an important characteristic feature of the PDF in Eq. (3).

In figure (3) we show a realization of a one-dimensional LWCP for which both the velocity PDF

$$F(|v|) \sim |v|^{-3/2}$$

and the collision time PDF $q(\tau)$, Eq. (3), decay slowly and are characterized by heavy tails. Here we observe, in addition to the long time intervals, in which no collision takes place, also rare events in which the velocity of the particle becomes very large. The longer we observe the process the longer are the free time intervals and higher are the velocities that are observed. Generally we expect a cutoff in the velocity spectrum and then the algebraic decay is not valid for very high velocities.
Fig. 2. A two state model where the velocity has the values ±1. The collision times are the same as found in figure (1).

The LWCP and the CTRW process, are intimately related as can be seen by considering realizations of the two processes. A CTRW process is defined through pausing times $\tau^p_i$ ($i = 1, 2, \cdots$) and jump lengths $x_i$, and a sequence of these pausing time and jump lengths is characterized by

$$\{[\tau^p_1, x_1], \cdots [\tau^p_n, x_n], \cdots [\tau^p_{N-1}, x_{N-1}], [\tau^p_N]\}.$$

The pausing times satisfy $\sum_{n=1}^{N} \tau^p_n = t$, with $t$ being the observation time and the location of the random walker is $x(t) = \sum_{n=1}^{N-1} x_n$. Notice that the $N$-th pausing time $\tau^p_N$ is not related to a displacement $x_N$. The CTRW is a jump process and therefore at the time of observation $t$ the particle is trapped motionless somewhere in the system. This is not the case for the LWCP where the time of free motion, $\tau^f_i$, is related to a displacement $x_i$. In this case a sequence is characterized by

$$\{[\tau^f_1, x_1], \cdots [\tau^f_n, x_n], \cdots [\tau^f_{N}, x_N]\}.$$
Fig. 3. A LWCP for which both the times between collisions and the velocities are described by PDFs with long tails. The LWCP is characterized by long time intervals in which no collision events take place, as well by events in which the particle gains very high velocities. Notice that we present $\ln(|v|)$ vs time. As discussed in the text, for such a process the mean squared displacement diverges.

The total displacement is $x(t) = \sum_{n=1}^{N} x_n$, and, unlike the CTRW, the summation includes the $N$-th term. One might expect that for a proper choice of coupling and PDFs the same asymptotic behavior will be found for both processes. Indeed, as expected, we find that when diffusion is normal, $\beta = 1$ in Eq. (1), the two pictures converge for long times. However, for systems exhibiting anomalous diffusion we find differences between the two approaches thus emphasizing the importance of investigating the LWCP.

The paper is organized as follows. In section (2) a solution of the model, in Fourier-Laplace space, is derived. Then, the role of initial conditions is considered in section (3). The influence of the first waiting time PDF and of the initial distribution of the velocities on the process are investigated. In section (4) the asymptotic behavior of our model is investigated. Depending on the parameters
of the model we find sublinear, enhanced and diverging mean squared displacements. A comparison between the new results, the coupled CTRW, Eq. (2), and other velocity approaches is given in section (5).

2 The Lévy Walk Collision Process (LWCP)

Let \( q(\tau) \) be the PDF of the independent time intervals between strong collision events. The survival probability

\[
W(t) = 1 - \int_0^t q(\tau) d\tau, \tag{5}
\]

is the probability that no collision event has taken place in the time interval (0, t). Let \( F(v) \) be the PDF of the velocity \( v \) of the particle. Between collision events the particle moves freely according to the law [see Eq. (2) and [28, 30]]

\[
r(t) = r(0) + vt^\nu \tag{6}
\]

with \( \nu \geq 0 \). When \( \nu = 0 \) the generalized velocity \( v \) is a displacement. Notice that only when \( \nu = 1 \), \( v \) has the dimensions of [length/time]. After each free motion event the particle’s velocity is resampled from the PDF \( F(v) \) and the LWCP is renewed.

We label the collision events in the interval (0, t) according to \( \{1, 2, \ldots s, \ldots\} \), and define \( \eta_s(r, t) \, dr \, dt \) as the probability that the s collision event takes place in the interval \( (r, r + dr) \) during the time interval \( (t, t + dt) \). The PDF \( \eta_s(r, t) \) is normalized

\[
\int_0^\infty dt \int dr \eta_s(r, t) = 1, \tag{7}
\]

where the spatial integration is over the whole space. The PDF \( \eta_s(r, t) \) satisfies the recursion relation

\[
\eta_s(r, t) = \int dv \int_0^t d\tau \eta_{s-1} (r - v\tau^\nu, t - \tau) \, F(v) \, q(\tau) \tag{8}
\]

for \( s \geq 1 \). The initial condition of starting from \( r = 0 \) at \( t = 0 \) is incorporated by the condition

\[
\eta_0(r, t) = \delta(r) \delta(t). \tag{9}
\]

The PDF \( \eta_s(r, t) \) is related to the PDF \( \rho(r, t) \) according to

\[
\rho(r, t) = \sum_{s=0}^\infty \int dv \int_0^t d\tau \eta_s (r - v\tau^\nu, t - \tau) \, F(v) \, W(\tau). \tag{10}
\]

We now introduce for the Fourier-Laplace transforms the covention that the arguments of a function indicate in which space the function is defined, e.g.
\( \rho (k, u) \) is the Fourier-Laplace of \( \rho (r, t) \). Using the convolution theorem, Eq. (8) can be written as:

\[
\eta_s (k, u) = \overline{q} (k, u) \eta_{s-1} (k, u),
\]

(11)

for \( s \geq 1 \), and \( \eta_0 (k, u) = 1 \). In Eq. (11) we have used the definition

\[
\overline{q} (k, u) \equiv L \left[ \tilde{F} (kr') q (\tau) \right],
\]

(12)

where the operator \( L \) is the Laplace transformation, and

\[
\tilde{F} (kr') \equiv \int dv e^{ik \cdot v \tau'} F (v),
\]

(13)

is the \( v \to kr' \) Fourier transform of \( F (v) \). Reverting Eq. (10) to the Fourier-Laplace space we find

\[
\rho (k, u) = \sum_{s=0}^{\infty} \overline{W} (k, u) \eta_s (k, u),
\]

(14)

where \( \overline{W} (k, u) \) is defined by Eq. (12), where \( q (\tau) \) is replaced by \( W (\tau) \). Using Eq. (11), which implies \( \eta_s (k, u) = \overline{q}^s (k, u) \), we find:

\[
\rho (k, u) = \frac{\overline{W} (k, u)}{1 - \overline{q} (k, u)}.
\]

(15)

It is easy to show that \( \rho (k = 0, u) = 1/u \), as it should from the normalization condition. Let as also note that Eq. (15) can be derived from a generalized master equation (in analogy to [30, 42])

\[
\frac{\partial \rho (r, t)}{\partial t} = \int dr' \int_0^t d\tau K (r - r', t - \tau) \rho (r', \tau')
\]

(16)

with the memory kernel

\[
K (k, u) = \frac{u \overline{W} (k, u) - 1 + \overline{q} (k, u)}{\overline{W} (k, u)}.
\]

(17)

To derive Eq. (17) one should Fourier-Laplace transform Eq. (16) and compare the result with the solution of our model, Eq. (15).
3 The Role of Initial Conditions

3.1 Other Renewal Processes

The choice of the initial condition can play a significant role in determining the nature of the anomalous transport [43]. In some cases it is important to distinguish between the PDF $h(t_1)$ of the time which elapses between the start of observation $t = 0$ and the first collision event at $t_1$, and the PDF $q(\tau)$. In our derivation of Eq. (15) we have assumed $h(t_1) = q(t_1)$. However, one may encounter situations where $h(t_1) \neq q(t_1)$. An example is the equilibrium renewal process [43] where

$$h_{eq}(t_1) = \frac{1 - \int_0^{t_1} q(\tau) d\tau}{\langle \tau \rangle}, \quad (18)$$

and the mean time between collisions $\langle \tau \rangle = \int_0^{\infty} \tau q(\tau) d\tau$ is assumed to be finite.

Another issue we consider here is the initial distribution of velocities. We denote by $P_0(v,0)$ the PDF of the velocity of the particles at time $t = 0$; generally $P_0(v,0) \neq F(v)$. While for a normal process initial conditions usually decay exponentially with time, for some LWCPs exhibiting anomalous diffusion the initial condition decays as a power law. As for normal transport systems it is of interest to find the relaxation patterns of the velocity PDF, and relate them to the fluctuations characterized, for example, by $\langle r^2(t) \rangle$ derived in a following section.

To calculate $\rho(k,u)$ we first define the survival probability

$$Z(t) = 1 - \int_0^t h(\tau) d\tau \quad (19)$$

which is the probability that no collision event has taken place from the start of the observation, at $t = 0$, to time $t$. Then, similarly to Eq. (15) we find that

$$\rho(k,u) = \overline{Z}_0(k,u) + \overline{h}_0(k,u) \overline{W}(k,u) \overline{1 - q(k,u)} \quad (20)$$

where the functions $\overline{Z}_0(k,u)$ and $\overline{h}_0(k,u)$ are defined according to:

$$\overline{f}_0(k,u) \equiv L \left[f(\tau) \tilde{P}_0(k\tau',0)\right]. \quad (21)$$

When $P_0(v,0) = F(v)$ and $h(\tau) = q(\tau)$, Eq. (20) reduces to Eq. (15).

3.2 The Time Dependence of the Velocity PDF

We define $P(v,t)$ to be the PDF to find the particle at time $t$ with a velocity $v$, assuming an initial condition $P_0(v,0)$. Since $Z(t)$, Eq. (19), is the probability that no collision event has taken place in $(0,t)$, and since a single collision event is needed to relax $P(v,t)$ to the equilibrium PDF $F(v)$, we obtain

$$P(v,t) = Z(t) P_0(v,0) + [1 - Z(t)] F(v). \quad (22)$$
The PDF \( P(v,t) \) satisfies the differential equation:

\[
\frac{\partial P(v,t)}{\partial t} = [P(v,t) - F(v)] \frac{\partial}{\partial t} \ln[Z(t)].
\]  
(23)

Assuming that the process is described by a rate \( \alpha \), so that \( h(t_1) = \alpha e^{-\alpha t_1} \) we have

\[
P(v,t) = \exp(-\alpha t) P_0(v,0) + [1 - \exp(-\alpha t)] F(v).
\]  
(24)

For this case the process is described by

\[
\frac{\partial P(v,t)}{\partial t} = -\alpha [P(v,t) - F(v)].
\]
(25)

This Poissonian process is identical to the strong collision model [38, 39]. Comparing Eq. (23) and Eq. (25) we realize that our strong collision model, Eq. (23) is similar to the standard strong collision model, Eq. (25), with a time dependent rate, determined by the transformation

\[
\alpha \rightarrow -\frac{\partial}{\partial t} \ln[Z(t)].
\]  
(26)

Characterizing anomalous transport processes by time dependent transport coefficients is well known but should be used with care as pointed out in [44]. Here we have been able to justify this approach for the evolution of \( P(v,t) \).

It is also possible to describe the process using an integro-differential equation

\[
\frac{\partial P(v,t)}{\partial t} = \int dv' \int_0^t d\tau K(v - v',t - \tau) P(v,\tau).
\]  
(27)

Assuming an initial condition \( P(v,t = 0) = \delta(v - v_0) \), the memory kernel in Fourier-Laplace space \( (v,t) \rightarrow (l,u) \) is found by an approach similar to that used to derive Eq. (17),

\[
K(l,u) = \frac{uh(u)[F(l) - e^{i l \cdot v_0}]}{e^{i l \cdot v_0} + h(u)[F(l) - e^{i l \cdot v_0}]}.
\]  
(28)

The solution of the nonlocal integro-differential, Eq. (27), and the differential equation Eq. (23), which is local in time, is Eq. (22). Thus, we see that the two approaches can be used to describe the anomalous process.
4 The Asymptotic Behavior of $\rho (k, u)$

We investigate the small $(k, u)$ behavior of $\rho (k, u)$, Eq. (15), for different $F(v)$ and $q(\tau)$. We consider functions which satisfy $F(v) = F(v)$ and so $\tilde{F}(k\tau) = \tilde{F}(k\tau)$, which yields $\rho (k, u) = \rho (k, u)$. Such a choice means that we have translational symmetry (i.e. $(v) = 0$) and so there is no net drift. We choose the PDFs which behave according to:

$$\tilde{F}(k\tau) \sim 1 - C_1 k^\delta \tau^{\delta \nu} \quad 0 < \delta \leq 2$$  \hspace{1cm} (29)

where $(k\tau)$ is small. For $\delta < 2$ the variance of $F(v)$ diverges. We also choose:

$$q(\tau) \sim \tau^{-(1+\gamma)},$$  \hspace{1cm} (30)

valid for large $\tau$. For $0 < \gamma < 1$ the mean time between collisions diverges, while when $1 < \gamma < 2$, $(\tau)$ is finite, all other integer moments of $q(\tau)$ diverge. The results which follow below will be compared with the coupled CTRW of Ref. [30].

The small $(k, u)$ behavior of $\rho (k, u)$ is determined in the following way. We first expand the denominator and nominator of Eq. (15) in the small parameter $k$,

$$\rho (k, u) \sim \frac{1}{u} \left[ \frac{1 - C_1 k^\delta f_1(u)}{1 + C_1 k^\delta f_2(u)} \right]$$  \hspace{1cm} (31)

with

$$f_1(u) = u \frac{L[\tau^{\delta \nu} W(\tau)]}{1 - q(u)}$$  \hspace{1cm} (32)

and

$$f_2(u) = \frac{L[\tau^{\delta \nu} q(\tau)]}{1 - q(u)}.$$  \hspace{1cm} (33)

We then analyze the small $u$ behavior of $f_1(u)$ and $f_2(u)$. Two important parameters $\delta^* \equiv \delta \nu$ and $\gamma$ control the asymptotic behavior of these functions. Notice also that for $\delta < 2$ Eq. (31) implies that for $t > 0$, the mean squared displacement diverges, this is expected since the variance of the velocity PDF $F(v)$ diverges.

We consider now the case when $1 < \gamma < 2$. Expanding Eqs. (32) and (33) to the lowest order of approximation in $u$, we find

$$f_1(u) \sim \begin{cases} u^{\gamma - \delta^* - 1} B_1^{\gamma^*} / C_1 & \gamma - 1 < \delta^* \\ B_1^{\gamma^*} / C_1 & \delta^* < \gamma - 1, \end{cases}$$  \hspace{1cm} (34)

and

$$f_2(u) \sim \begin{cases} u^{\gamma - \delta^* - 1} B_2^{\gamma^*} / C_1 & \gamma < \delta^* \\ u^{-1} B_2^{\gamma^*} / C_1 & \gamma > \delta^*. \end{cases}$$  \hspace{1cm} (35)
Here $B_i^{\gamma \delta^*}$, with $i = 1, 2$, are coefficients determined from Eqs. (32) and (33) for a given $q(\tau)$. Notice that for $\gamma < \delta^*$ the power law behavior of $f_1(u)$ and $f_2(u)$ is given by the same exponent $\gamma - \delta^* + 1$. This means that for this case both functions are needed in order to obtain the correct asymptotic behavior of $\rho(k,u)$. Using Eqs. (31), (34) and (35) we find

$$
\rho(k,u) \sim \frac{u^{\delta^* - \gamma} - B_1^{\gamma \delta^*} k^\delta / u}{u^{\delta^* - \gamma + 1} + B_2^{\gamma \delta^*} k^\delta} \quad \gamma < \delta^*
$$

$$
\rho(k,u) \sim \frac{1 - B_1^{\gamma \delta^*} k^\delta u^{-\delta^* + \gamma - 1}}{u + B_2^{\gamma \delta^*} k^\delta} \quad \delta^* < \gamma < \delta^* + 1
$$

$$
\rho(k,u) \sim \frac{1 - B_1^{\gamma \delta^*} k^\delta}{u + B_2^{\gamma \delta^*} k^\delta} \quad \delta^* + 1 < \gamma
$$

(36)

When $\delta^* + 1 < \gamma$ one can approximate

$$
\rho(k,u) \sim \frac{1}{u + \left( B_1^{\gamma \delta^*} + B_2^{\gamma \delta^*} \right) k^\delta}
$$

(37)

which, when inverted Laplace transformed, yields the familiar Lévy PDF

$$
\rho(k,t) \sim \exp\left[ -t \left( B_1^{\gamma \delta^*} + B_2^{\gamma \delta^*} \right) k^\delta \right].
$$

(38)

When $\delta = 2$, the mean squared displacement is finite. From Eq. (31) it is easy to show that

$$
\langle r^2(u) \rangle = \frac{2C_1}{u} [f_1(u) + f_2(u)].
$$

(39)

Using this equation, the large $t$ behavior of the mean squared displacement is then found to be

$$
\langle r^2(t) \rangle \sim \begin{cases} 
2 \frac{B_1^{2\nu} + B_2^{2\nu}}{t(2\nu - \gamma + 2)} t^{2\nu - \gamma + 1} & \gamma < 2\nu \\
2B_2^{2\nu} t & \gamma > 2\nu,
\end{cases}
$$

(40)

for $1 < \gamma < 2$. When $\gamma < 2\nu$ the diffusion is enhanced and when $2\nu < \gamma$ the diffusion is normal.

We now find the small $(k, u)$ behavior when $0 < \gamma < 1$. Expanding Eq. (32)

$$
f_1(u) \sim \begin{cases} 
u^{1-\gamma} B_1^{\gamma \delta^*} / C_1 & \delta^* < \gamma - 1 \\
u^{-\delta^*} B_1^{\gamma \delta^*} / C_1 & \gamma - 1 < \delta^*,
\end{cases}
$$

(41)

and from Eq. (33)

$$
f_2(u) \sim \begin{cases} 
u^{-\delta^*} B_2^{\gamma \delta^*} / C_1 & \gamma < \delta^* \\
u^{-\gamma} B_2^{\gamma \delta^*} / C_1 & \gamma > \delta^*.
\end{cases}
$$

(42)
Using Eq. (31) we find
\[ \rho (k, u) \sim \frac{u^{\gamma \xi - 1} - B_1^{\gamma \xi} k^\xi / u}{u^{\gamma \xi} + B_2^{\gamma \xi} k^\xi} \quad \gamma < \delta^* \]
\[ \rho (k, u) \sim \frac{u^{\gamma \xi - 1} - B_1^{\gamma \xi} k^\xi u^{-\delta^* + \gamma - 1}}{u^{\gamma \xi} + B_2^{\gamma \xi} k^\xi} \quad \delta^* < \gamma < \delta^* + 1 \]
\[ \rho (k, u) \sim \frac{u^{\gamma \xi - 1} - B_1^{\gamma \xi} k^\xi}{u^{\gamma \xi} + B_2^{\gamma \xi} k^\xi} \quad \delta^* + 1 < \gamma \] (43)

For \( \gamma < \delta^* \) the exponents in Eq. (43) are independent of \( \gamma \) and for \( \delta = 2 \)
\[ \langle r^2 (t) \rangle \sim \begin{cases} 
2 B_1^{\gamma + 2 \nu} + B_2^{\gamma + 2 \nu} t^{2 \nu} & \gamma < 2 \nu \\
2 B_2^{\gamma + 2 \nu} t^\gamma & \gamma > 2 \nu.
\end{cases} \] (44)

We see that when \( \gamma < 2 \nu \) the diffusion is enhanced for \( \nu > 1/2 \) and slow for \( \nu < 1/2 \). The regime \( \gamma > 2 \nu \) exhibits a subdiffusive behavior.

All these different types of behaviors of the mean squared displacement are summarized in figure (4). Such a phase diagram can be derived also from the coupled CTRW, Eq. (2). As far as we know this diagram was presented first in [30], where slightly different notations were used, and later by [28]. Other, related, though more complex, phase diagrams were found by Weeks et al [45] in the context of tracer diffusion in rotating flows. They have taken into account also the possibility of sticking events in their random walk scheme.

4.1 An Example

As an example consider the case \( \nu = 1 \) with Gaussian velocities,
\[ \tilde{F} (k \tau) = \exp (-k^2 \tau a^2 / 2). \] (45)

For \( q (\tau) \) we choose
\[ q (\tau) = \frac{\gamma}{(1 + \tau)^{\gamma + 1}}, \] (46)

which yields
\[ q (u) \sim \begin{cases} 
1 - \Gamma (1 - \gamma) u^\gamma & 0 < \gamma < 1 \\
1 - (\tau) u - \Gamma (1 - \gamma) u^\gamma & 1 < \gamma < 2
\end{cases} \] (47)

with \( \langle \tau \rangle = 1 / (\gamma - 1) \). A simple calculation, using Eqs. (31)-(33), shows
\[ \rho (k, u) \sim \frac{u - 0.5a^2 (1 - \gamma)(2 - \gamma)k^2 u^{-2}}{u^2 + 0.5a^2 (1 - \gamma)\gamma k^2} \] (48)
Fig. 4. The phase diagram, for $\delta = 2$, showing the different types of behaviors: (a) N for normal diffusion (b) E for enhanced diffusion and (c) S for sublinear, dispersive, diffusion. The different types of behaviors are specified in Eqs. (40) and (44).

for $0 < \gamma < 1$, and

$$\rho(k, u) \sim \frac{u^{2-\gamma} - 0.5a^2(\gamma - 1)\Gamma(3-\gamma)k^2/u}{u^{3-\gamma} + 0.5a^2\gamma(\gamma - 1)\Gamma(2-\gamma)\gamma k^2}$$

(49)

for $1 < \gamma < 2$. Using Eqs. (48)-(49), the mean squared displacement is:

$$\langle r^2(t) \rangle \sim \begin{cases} a^2(1-\gamma)t^2 & 0 < \gamma < 1 \\ 2a^2\frac{(\gamma-1)}{(2-\gamma)(3-\gamma)}t^{3-\gamma} & 1 < \gamma < 2. \end{cases}$$

(50)

The regime $0 < \gamma < 1$ is called the ballistic regime. The behavior $\langle r^2 \rangle \sim t^{3-\gamma}$ in Eq. (50), has been recently reported in a large number of systems [4]. When $\gamma > 2$ diffusion is normal and $\langle r^2 \rangle \sim t$. 
5 CTRW vs the Lévy Walk Collision Process

A comparison is made between the results obtained here and those obtained using the framework of the coupled and uncoupled jump CTRW [21, 20]. First consider $\nu = 0$, which according to Eq. (6) implies that the generalized velocity $v_{\nu=0}$ is in fact a displacement. Since the displacement at each step is statistically independent of the time interval between steps, it is not surprising that for $\nu = 0$ our Lévy walk collision model is similar to the decoupled version of the CTRW. To see this, note that from Eq. (12)

$$\bar{q}_{\nu=0}(k, u) = \bar{W}(k, u) q(u)$$ (51)

and $\bar{W}_{\nu=0} = \bar{F}(k) \bar{W}(u)$. From Eq. (15) we find

$$\rho_{\nu=0}(k, u) = \frac{1 - q(u)}{u} \frac{\bar{F}(k)}{1 - \bar{F}(k) q(u)}.$$ (52)

For the decoupled version of the CTRW where $\psi(r, t) = F(r) q(t)$ one finds [20, 21, 30]

$$\rho_{ct}(k, u) = \frac{1 - q(u)}{u} \frac{1}{1 - \bar{F}(k) q(u)}.$$ (53)

Eqs. (52) and (53) clearly differ. The factor $\bar{F}(k)$ which appears in our collision model results from the free evolution in the time interval between the last collision event in the sequence and the observation time $t$. For the CTRW the jumping random walker, is fixed in a "deep trap" during this time interval. This seemingly small difference in the models can become important as we demonstrate.

We compare now the transformed probability density $\rho(k, u)$, Eq. (15), and the one obtained within the framework of the coupled CTRW. According to Eq. (21) in Ref. [30] the CTRW result for the coupled kernel, Eq. (2), is

$$\rho_{ct}(k, u) = \frac{1 - \psi(u)}{u} \frac{1}{1 - \psi(k, u)}$$ (54)

and $\psi(u) = \psi(k = 0, u)$. We rewrite Eq. (15) as

$$\rho(k, u) = \frac{1 - q(u)}{u} \frac{1}{1 - \bar{q}(k, u)} + \frac{\bar{W}(k, u) - \bar{W}(0, u)}{1 - \bar{q}(k, u)}$$ (55)

where

$$\bar{W}(0, u) = \frac{1 - q(u)}{u}.$$ (56)

The two models can be compared to each other if we identify

$$\bar{q}(k, u) = \psi(k, u)$$ (57)
and hence \( \bar{q}(k = 0, u) = q(u) = \psi(u) \). To quantify the difference between the two models we define:

\[
\Delta \equiv \rho(k, u) - \rho_{ct}(k, u) = \frac{\bar{W}(k, u) - \bar{W}(0, u)}{1 - \bar{q}(k, u)}.
\] (58)

For \( k = 0 \) we have \( \Delta = 0 \), as expected from the normalization condition.

Let us examine now the difference \( \Delta(k, u) \) using the small \( k \) expansion Eq. (29),

\[
\Delta(k, u) \sim -C_1 \frac{k^\delta}{u} f_1(u),
\] (59)

where \( f_1(u) \) has been defined already in Eq. (32). The CTRW result can be written in our notation using Eqs. (31), (58) and (59)

\[
\rho_{ct}(k, u) \sim \frac{1}{u} \left[ \frac{1}{1 + C_1 k^\delta f_2(u)} \right].
\] (60)

Hence, all our results reduce to the CTRW results when we assign \( f_1(u) = 0 \) in Eqs. (31) and (39), which yields \( B_1^{\delta^*} = 0 \) in Eqs. (36) and (43). The question remains whether one can approximate \( \rho_{ct}(k, u) \) by \( \rho(k, u) \). Comparing Eq. (31) and Eq. (60) one reaches the conclusion that only if

\[
f_1(u) \ll f_2(u)
\] (61)

such an approximation is justified. When all moments of \( q(\tau) \) exist [e.g. when \( q(\tau) = \alpha \exp(-\alpha \tau) \)] then Eq. (61) is satisfied when \( u \) is small, meaning that \( \rho(r, t) \approx \rho_{ct}(r, t) \) for large \( t \). However, comparing Eq. (34) with (35) and Eq. (41) with (42) we see that the condition in Eq. (61) is not always satisfied. This means that the contribution from \( f_1(u) \) cannot be neglected and both \( f_1(u) \) and \( f_2(u) \) determine the long time behavior of \( \rho(r, t) \). For these cases one cannot approximate the CTRW PDF \( \rho_{ct}(r, t) \) with the LWCP PDF \( \rho(r, t) \) even for long times.

The difference between the two results can be easily understood when \( \delta = 2 \). Then \( \langle r^2 \rangle \) is non diverging and

\[
\langle r^2(t) \rangle - \langle r^2(t) \rangle_{ct} = L^{-1} \left[ -\frac{\partial^2 \Delta(k, u)}{\partial k^2} \right]_{k=0},
\] (62)

where \( L^{-1} \) is the inverse Laplace operator. We find, using Eq. (40),

\[
\langle r^2 \rangle - \langle r^2 \rangle_{ct} \sim \begin{cases} 2 \frac{B_{2}^{2\nu}}{I(2\nu-\gamma+3)} t^{2\nu-\gamma+1} & \gamma < 2\nu \\ 0 & \gamma > 2\nu, \end{cases}
\] (63)

when \( 1 < \gamma < 2 \), and from Eq. (44)

\[
\langle r^2 \rangle - \langle r^2 \rangle_{ct} \sim \begin{cases} 2 \frac{B_{2}^{2\nu}}{I(2\nu+1)} t^{2\nu} & \gamma < 2\nu \\ 0 & \gamma > 2\nu, \end{cases}
\] (64)
when $0 < \gamma < 1$. We see that the LWCP is more "efficient" then the CTRW process (i.e. $\langle r^2 \rangle - \langle r^2 \rangle_{ct} \geq 0$), due to a considerable difference in the prefactors of the two processes. Such a behavior, is very different from ordinary diffusion processes where random walks provide a good approximation to simple collision models.

Even though the deviations between the two processes exist we emphasize that the exponents appearing in the long time limit of the mean squared displacement for the CTRW and the LWCP are identical once the correspondence Eq. (57) is made. In other words, the non vanishing exponents appearing in Eqs. (63) and (64) are identical to those in Eqs. (40) and (44). Thus the difference between the two processes is characterized by different prefactors which are the cause for our finding $\langle r^2 \rangle - \langle r^2 \rangle_{ct} \geq 0$ for long times.

The deviations between the LWCP and the coupled jump version of the CTRW can be understood based upon the following argument. For the CTRW at the observation time $t$ the random walker is trapped in a lattice point. The random walker has occupied this trap for a time $t - t_i^f$, where $t_i^f$ is the location on the time axis at which the last jump in the sequence of jumps was executed. On the other hand, for the LWCP the particle is always moving according to the law in Eq. (6). This evolution includes the time interval $t - t_i$ and here $t_i$ is the location on the time axis at which the last collision in the sequence has occurred. To see this better, assume that for the LWCP during the last time interval in the sequence the particle does not evolve but rather stays fixed at the location of the last collision event in the sequence occurred. Then we have to replace Eq. (10) with

$$
\rho (r, t) = 
\sum_{s = 0}^{\infty} \int_0^t d\tau \eta_s (r, t - \tau) W (\tau),
$$

leaving Eqs. (5)-(9) unchanged. Then, following the same procedure we have followed to derive Eq. (15), we derive the result:

$$
\rho (k, u) = \frac{W (u)}{1 - \tilde{q} (k, u)}
$$

which is the CTRW result, Eq. (53), provided that the condition in Eq. (57) is satisfied. For normal systems the additional evolution in the last interval in the sequence does not contribute significantly to $\rho (r, t)$ when $t$ is large, and so the collision process and the CTRW give practically the same results. However, if the last interval in the sequence is very long, in an averaged sense, then the difference between the coupled CTRW process and the LWCP may become large.

As mentioned in the Introduction, our generalized approach maps onto the previous approach when the magnitude of the velocity is a constant $|v| = 1$, and the collisions change only the directions of motion. We summarize now similarities and differences between the results obtained within these two frameworks. First, when the variance of $F (v)$ diverges there is no place for comparison, since
for the previous theories \( \langle r^2 \rangle \) is finite while our approach yields a diverging \( \langle r^2 \rangle \). When the first two moments of \( F(v) \) exist, the exponents controlling the diffusion in Eqs. (40) and (44) are identical to those obtained previously [25, 28] while the prefactors are different.

Another difference between the LWCP and the other constant velocity approaches concerns the wings of the PDF \( \rho(r,t) \). For the case \(|v| = 1 \) one finds that

\[
\rho(r,t) = 0 \quad \text{when} \quad |r| > t. \tag{67}
\]

This result is obvious, since if the particle has a maximal speed there is probability zero to find it beyond \(|v|t\). What is found for instance in Ref. [46], is that delta peaks appear in the solution for \( \rho(r,t) \), at \(|r| = t \). Now, if we choose \( F(v) \) to be a Gaussian, Eq. (67) is not valid since the probability of finding the particle with a velocity \( v < \infty \) is nonzero. The delta peaks are not expected for this \( F(v) \). To see this we define

\[
H(r,t) \equiv \int dv \int_0^t d\tau \delta (r - vr^\nu) \delta (t - \tau) F(v) W(\tau). \tag{68}
\]

The function \( H(r,t) \), is the \( s = 0 \) term in the sum that appears on the right hand side of Eq. (10). It describes the contribution to \( \rho(r,t) \) from trajectories for which the particle did not encounter collisions. We consider the one-dimensional case with \( \nu = 1 \) and a Gaussian \( F(v) \). Then

\[
H(x,t) = \frac{W(t)}{\sqrt{2\pi t}} \exp \left[ - \left( \frac{x^2}{2t^2} \right) \right]. \tag{69}
\]

While for the two state model, \( F(v) = 0.5 [\delta (v + v_0) + \delta (v - v_0)] \), then

\[
H(x,t) = \frac{W(t)}{t} 0.5 [\delta (x/t + v_0) + \delta (x/t - v_0)]. \tag{70}
\]

We see that \( H(x,t) \) behaves differently for the two choices of the PDF \( F(v) \). For the Gaussian process it is centered around \( x = 0 \), while for the constant velocity approach, delta peaks appear at \( x = \pm |v|t \). Thus, the choice of \( F(v) \) has an influence on the asymptotic shape of \( \rho(r,t) \).

6 Summary

In this work we have investigated a strong collision model which we have called the Lévy walk collision process (LWCP). The LWCP scheme can be viewed as a generalization of the CTRW for the case when the velocities of the random walkers are randomly distributed. An extension of the normal Brownian motion to anomalous diffusion with sublinear, enhanced or diverging diffusion has been given. The CTRW framework with coupled kernels also results in such diffusional patterns. However the CTRW, describing a jump process, considers positions of
the random walkers rather then their velocities and therefore it is not suited to
describe the situation we are interested in. Furthermore, we have shown that
differences exist between the CTRW and our results. Thus, even though the
exponents and the dynamical phase diagram of the two models are the same,
the two approaches are nonidentical. The LWCP is found to be more "efficient"
then the CTRW.

Unlike the previous velocity models, our work is not restricted to the condi-
tion that \(|v|\) is a constant, rather velocities are random variables described by
the PDF \(F(v)\). Our model allows to consider the case when \(F(v)\) is long-tailed.
Even for the case when the variance of \(F(v)\) is finite the asymptotic behavior
of \(\rho(r, t)\) behaves differently for the two approaches.

Here we have considered in some detail the case where \(F(v)\) is symmetric
with a zero mean. In general one can include the case where the mean velocity
is finite and then an anomalous drift is expected. We shall discuss this drift in
the velocity field \(F(v)\) in a future publication.

The LWCP assumes strong collisions, which means that there are no corre-
lations between the velocity of a particle just before and just after a collision
event. A Gaussian one-dimensional model with either weak or strong collisions
has been investigated recently by one of the authors in Refs. [47, 48].

Since our model considers the random distribution of velocities, one may
follow the evolution the velocity PDF \(P(v, t)\) to the equilibrium \(F(v)\). We
have shown that \(P(v, t)\) can be derived from a differential equation with a
time dependent relaxation coefficient \(\alpha(t)\). This equation generalizes the strong
collision model which assumes an exponential process and which has been used
frequently in different fields. We believe our approach will find its applications
for non-Gaussian diffusion processes.

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Appendix A

Ways to generate random variables described by different types of PDFs (e.g.
the exponential, the Lorentzian and the Gaussian PDFs) can be found in Ref.
[49]. Here we show how to generate random variables whose PDF follows the
rule

\[ q(\tau) \sim \tau^{-(1+\varepsilon)} \quad (71) \]

for \(\tau \to \infty, 0 < \tau < \infty\). We have in mind cases where the exact behavior of the
PDF for small \(\tau\) is irrelevant. Two main methods to generate random variables
are usually used [49] (a) an accept-reject method, which is not efficient and
(b) a transformation method. Here we give a simple transformation rule which
generates random variable described by a longed tailed PDF.

We use a random number generator which generates a random variable \(u\)
which is distributed uniformly in the interval \(0 < u < 1\). Then we define the
transformation (for \(\xi > 0\))

\[ \tau = \left[ \tan \left( \frac{u \pi}{2} \right) \right]^\xi \quad (72) \]
and hence
\[
q(\tau) = p(u)\left| \frac{du}{d\tau} \right| = \left( \frac{2}{\pi \xi} \right)^{(1-\xi)/\xi} \frac{\tau^{(1-\xi)/\xi}}{(1 + \tau^{2/\xi})}
\]  \hspace{1cm} (73)

and \( p(u) \) is the uniform PDF. For long times we have
\[
q(\tau) \sim \left( \frac{2}{\pi \xi} \right) \tau^{-(1+1/\xi)}
\]  \hspace{1cm} (74)

and hence if we identify \( 1/\xi = z \) our goal is accomplished.

References

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