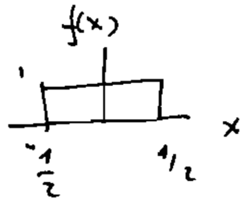
LAquila 2015

Gauss-Lévy Central limit theorem:

- $f(x)\,dx$ Prob. of having random variable x in the interval $(x, x+dx)$.

- $f(x) \geq 0$, $\int_{-\infty}^{\infty} f(x)\,dx = 1$ normalization.



- $f(x) = f(-x)$ for simplicity

$\langle x \rangle = 0$, $\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

- Lorentzian PDF $f(x) = \frac{1}{\pi} \frac{a}{a^2+x^2}$

x^{-2}

fat tailed!

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \underbrace{\frac{x^2}{\pi} \frac{a}{a^2+x^2}}_{\to_{x\to\infty} \Rightarrow \frac{a}{\pi}} dx = \infty$$

Characteristic function

$$\langle e^{ikx} \rangle = \int_{-\infty}^{\infty} f(x) e^{ikx}\,dx$$

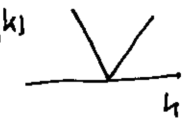
	$f(x)$	$\langle e^{ikx} \rangle$	
finite variance	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$	$e^{-k^2\sigma^2/2} \sim 1 - \frac{k^2\sigma^2}{2}$	analytical
	$\frac{1}{\pi} \frac{a}{x^2+a^2}$	$e^{-\lvert k \rvert a}$	NON analytical

$$\langle e^{ikx}\rangle = \int_{-\infty}^{\infty} f(x) \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!} dx = 1 - \frac{k^2\langle x^2\rangle}{2} + \ldots\ldots$$

$$\langle x^2\rangle = -\frac{d^2}{dk^2}\langle e^{ikx}\rangle\Big|_{k=0}$$

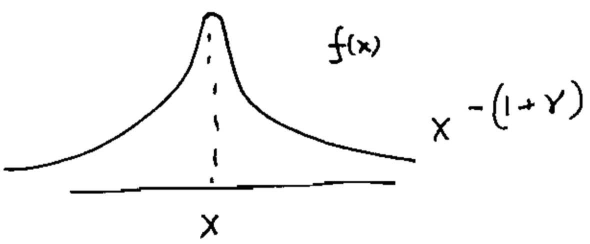
Lorentzian

$$\langle x^2\rangle = -\frac{d^2}{dk^2} e^{-|k|a}\Big|_{k=0} = -\frac{d^2}{dk^2}\left[1 - |k|a + \ldots\right] = \infty$$



More generally

x space



k space

$$\langle e^{ikx}\rangle = 1 - \tilde{A}|k|^\gamma$$

$0 < \gamma <$

Lévy - Gauss CLT

Let $\{x_1 \ldots\ldots x_N\}$ be N I.I.D. R.V with a common PDF $f(x)$ (symmetric), $f(x) \sim x^{-(1+\gamma)}$ $0<\gamma<2$,

$\langle e^{ikx}\rangle = \tilde{f}(k) \sim 1 - \tilde{A}|k|^\gamma$ $k \to 0$ then:

Let $z = \frac{\sum x_i}{N^{1/\gamma}}$

$$\boxed{\int_{-\infty}^{\infty} e^{ikz} \underbrace{P(z)}_{\text{PDF of } z} dz \xrightarrow[N\to\infty]{} e^{-A|k|^\gamma}}$$

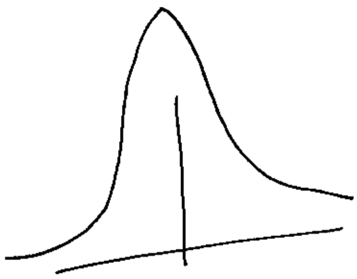
$$\langle e^{ikz}\rangle = \langle e^{ik\sum^{N} \frac{x_i}{N^{1/\gamma}}}\rangle = \langle e^{ikx_1/N^{1/\gamma}}\rangle \cdots \langle e^{ikx_N/N^{1/\gamma}}\rangle$$

$$= \Big[1 - \frac{A}{N}|k|^\gamma + \underbrace{\frac{\langle^2 k^2}{2N^{2/\gamma}}}_{\substack{\text{negligible}\\ \text{for } N\to\infty}} + \dots\Big]^N \longrightarrow e^{-\tilde{A}|k|^\gamma}$$

$e^{-|k|^\gamma} \;\circ\!\!-\!\!\bullet\; L_{\gamma,1,0}(x)$

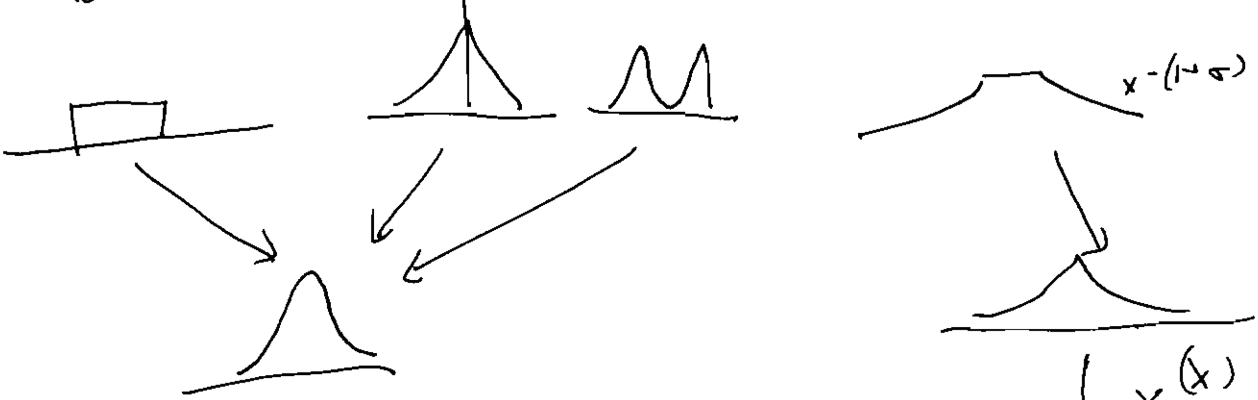
$\gamma = 2 \longleftrightarrow$ Gaussian

$\gamma = 1 \longleftrightarrow$ Lorentzian



$x^{-(1+\gamma)}$

SUM of IID R.V converges in distribution to either the Gaussian or Lévy distribution



$x^{-(1+\sigma)}$

Gauss

$L_\gamma(x)$

Levy

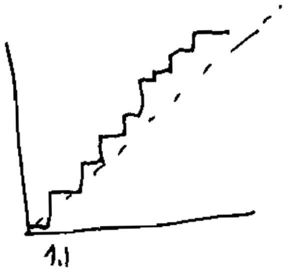
Generate fat tailed R.V on computer

$\underbrace{t}_{\text{R.V}} = \frac{1}{|1-x|^{1/\alpha}}$ $\quad 0 < x < 1$ uniformly distributed

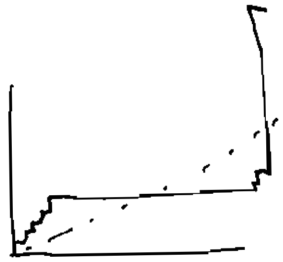
$$f(t) = \underbrace{p(x)}_{1}\left|\frac{dx}{dt}\right|$$

$$\boxed{f(t) = \alpha\, t^{-(1+\alpha)} \qquad 1 < t < \infty}$$

Stair case



$f(x) = e^{-x} \qquad x > 0$



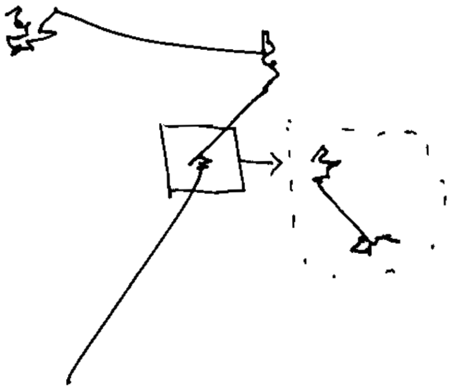
$f(x) \sim x^{-3/2}$

Largest x_i is of the order of the sum.

Lévy flight in 2D

$f(x) \sim x^{-(1+\gamma)}$

angle uniform



Brownian like motion

$f(x) \sim e^{-x} \qquad x > 0$

angle uniform

It is a fractal!

Mandelbrot

$$\langle e^{ik(x_1 + x_2)} \rangle = e^{-a|k| - a|k|} = e^{-2a|k|}$$

PDF of the sum is (up to factor ~~scale~~) the same as the individual step.

No characteristic scale (variance diverges).

$\boxed{\text{Sum looks like the component}}$

Scale (size of average jump) diverges