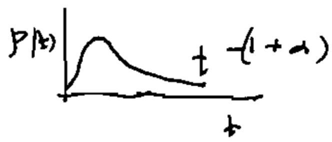
L'Aquila 2015

Tauberian theorem

Consider a one sided PDF $\quad t > 0$

$$p(t) \sim A\, t^{-(1+\alpha)} \qquad t \to \infty$$

$A > 0, \quad 0 < \alpha < 1$



Here $\langle t \rangle = \infty \qquad \int_0^\infty t\, p(t)\, dt = \infty.$

Investigate the small u behavior of the Laplace transform $\quad \hat{p}(u) = \int_0^\infty e^{-ut}\, p(t)\, dt.$

Example:

$$p(t) = \frac{1}{2\sqrt{\pi}}\, t^{-3/2} e^{-1/4t} \quad \longrightarrow \quad \hat{P}(u) = e^{-u^{1/2}}$$

$$p(u) \sim 1 - u^{1/2} \qquad u \to 0$$

$$P(t) \sim \frac{1}{2\sqrt{\pi}}\, t^{-3/2}$$

$$\int_0^\infty e^{-ut} p(t)\, dt = \int_0^{t_0} e^{-ut} p(t)\, dt + A \int_{t_0}^\infty t^{-(1+\alpha)} e^{-ut} dt + \epsilon$$

$\epsilon = 0$ if $\quad p(t) = A\, t^{-(1+\alpha)} \quad$ for $\quad t > t_0.$

Neglect ϵ.

$$= \underbrace{\int_0^{t_0} e^{-ut} p(t) dt + A \int_{t_0}^\infty t^{-(1+\alpha)} dt}_{1 \text{ when } u \to 0} + A \int_{t_0}^\infty t^{-(1+\alpha)} \left(e^{-ut} - 1\right) dt$$

Aside:

$$A \int_{t_0}^{\infty} t^{-(1+\alpha)} \left(e^{-ut}-1\right) dt = A \int_{ut_0}^{\infty} \left(\frac{x}{u}\right)^{-1-\alpha} \left(e^{-x}-1\right) \frac{dx}{u}$$

$ut = x$

$dt = \frac{dx}{u}$

$t = x/u$

$$= u^{\alpha} A \underbrace{\int_{ut_0}^{\infty} \frac{e^{-x}-1}{x^{1+\alpha}} dx}_{\substack{\Downarrow \\ u \to 0 \quad \Gamma(-\alpha)}} \quad (*)$$

$$\boxed{\begin{array}{l} \text{hence if} \quad p(t) \sim A t^{-(1+\alpha)} \quad 0 < \alpha < 1 \\ p(u) \sim 1 + A u^{\alpha} \Gamma(-\alpha) \end{array}}$$

$$\Gamma(-1/2) = -2\sqrt{\pi}$$

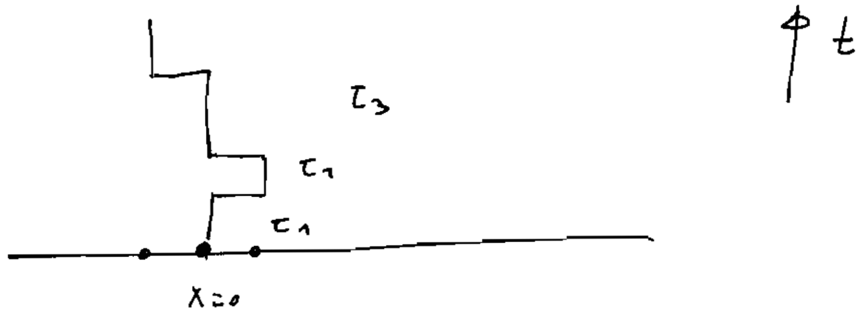
(*) here we neglect $u^{\alpha} A \int_0^{ut_0} \underbrace{\frac{e^{-x}-1}{x^{1+\alpha}}}_{x^{-\alpha}} dx \sim A u^{\alpha} x^{1-\alpha}\Big|_0^{ut_0}$

$\sim u$

hence small

Continuous Time Random Walk

Sher - Montroll 1975



1. Start on $x=0$.
2. wait on $x=0$ for time τ_1.
3. jump to $x=a$ or $x=-a$ equal prob.
4. Renew.

$\{\tau_1, \dots \tau_N \dots\}$ IID RV

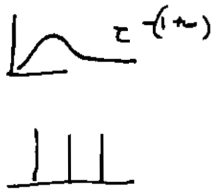
$\{\delta x_1 \dots \delta x_i \dots\}$ IID RV

PDF

$\psi(\tau)$

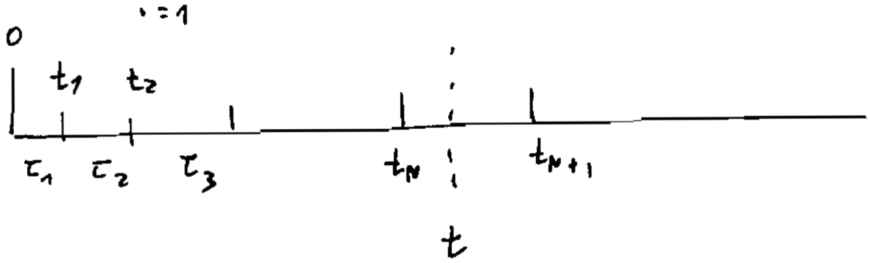
$f(x)$

Symmetric



$$\mathcal{X} = \sum_{i=1}^{N} \delta x_i$$

N R.V



no bias $\langle \delta x \rangle = 0$

$$\langle \mathcal{X}^2 \rangle = \langle N \rangle \underbrace{\langle (\delta x)^2 \rangle}_{\text{finite}}$$

$$\langle N \rangle \sim \frac{t}{\langle \tau \rangle}$$ normal case

$$\langle \mathcal{X}^2 \rangle = 2 \frac{\langle \delta x^2 \rangle}{2\langle \tau \rangle} t = 2D\, t$$ EINSTEIN

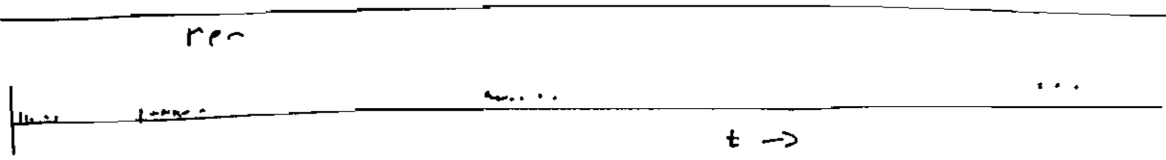
↳ Diffusion.

$$\langle \tau \rangle = \int_0^\infty \tau \, \psi(\tau) \, d\tau = \infty \qquad \psi(\tau) \sim \tau^{-(1+\alpha)}$$

$$\text{``}\langle \tau \rangle\text{''} \sim \int_0^t \tau^{-\alpha} \, d\tau \sim t^{1-\alpha}$$

$$\langle N \rangle \sim \frac{t}{\text{``}\langle \tau \rangle\text{''}} \sim t^{\alpha}$$

$$\langle X^2 \rangle \sim t^{\alpha} \qquad \text{sub-diffusion}$$



Prob $p_N(t)$ number of renewals in $(0, t)$

$$\hat{P}_N(u) = \left\langle \int_0^\infty \Theta(t_N < t < t_{N+1}) \, e^{-ut} \, dt \right\rangle$$

$$\Theta(t_N < t < t_{N+1}) = \begin{cases} 1 & \text{if } t_N < t < t_{N+1} \\ 0 & \text{otherwise} \end{cases}$$

$$P_N(u) = \left\langle \int_{t_n}^{t_{n+1}} e^{-ut} \, dt \right\rangle = \left\langle \frac{e^{-u t_n} - e^{-u t_{n+1}}}{u} \right\rangle = \frac{1 - \hat{\psi}(u)}{u} \, \hat{\psi}(u)^N$$

$$\langle e^{-u t_n} \rangle = \hat{\psi}(u)^n \qquad \text{since} \quad t_n = \sum_{i=1}^n \tau_i$$

Montroll Weiss Eq.

$$p(x,t) = \sum_{N=0}^{\infty} p(x,N)\; P_N(t)$$

$$p(x,N) \Leftrightarrow \tilde{f}^N(k) \Leftarrow \quad X = \sum \delta x_i$$

$$\hat{p}(k,u) = \sum_{N=0}^{\infty} \tilde{f}^N(k)\, \psi^N(u)\, \frac{1-\psi(u)}{u}$$

$$\boxed{p(k,u) = \frac{1-\psi(u)}{u\,[1-\tilde{f}(k)\,\psi(u)]}}$$

MSD

$$\langle X^2 \rangle = \langle \delta x^2 \rangle \langle N \rangle$$

$$\langle N \rangle = \sum_{N=0}^{\infty} N\, P_N(u) = \sum_0^{\infty} N\, \frac{1-\psi(u)}{u}\, \psi^N(u) = \frac{1-\psi(u)}{u}\, \psi(u) \sum_0^{\infty} \frac{d}{d\psi} \psi^N(u)$$

$$= \frac{1-\psi(u)}{u}\, \psi(u)\, \frac{d}{d\psi}\, \frac{1}{1-\psi}$$

$$\boxed{\langle N \rangle = \frac{\psi(u)}{u[1-\psi(u)]}}$$

$$\psi(u) \sim 1 - Au^{\alpha}$$

$$\langle N \rangle \sim \frac{1}{Au^{\alpha+1}}$$

$$\boxed{\langle N \rangle \sim \frac{1}{\Gamma(1+\alpha)}\,\frac{t^{\alpha}}{A}}$$

Summary

When $\psi(\tau) \sim A\, \tau^{-(1+\alpha)}$ $\qquad 0 < \alpha < 1$

$$\langle X^2 \rangle = 2 \underbrace{\frac{\langle \delta x^2 \rangle}{2\,\Gamma(1+\alpha)\, A}}_{D_\alpha} t^{\alpha}$$

$\alpha = 1$ recover Einsten relation.

Hence the tail of the PDF of waiting time (parameters α, A) control the anomalous diffusion and in this sense they replace the average $\langle \tau \rangle$ in Einstein's theory.

Disorder Induced Sub-diffusion

traps with depth $-E_i$ IID RV

$$\tau_i = \tau_0 e^{E_i / k_B T}$$ Arrenhius law

$$g(E) = \frac{1}{T_g} e^{-E/T_g}$$

WAIT JUMP MODEL

$$\psi(\tau) = g(E) \left| \frac{d \Delta E}{d \tau} \right|$$

$$E = T \ln \frac{\tau}{\tau_0}$$

$$\psi(\tau) = \frac{1}{T_g} \exp\left[- \frac{T}{T_g} \ln(\tau/\tau_0) \right] \frac{T}{\tau}$$

$$\boxed{\alpha = T / T_g}$$

$$\boxed{\psi(\tau) = \frac{T}{T_g} \tau^{-\left(1 + \frac{T}{T_g}\right)} \quad \tau \gg 1}$$

This is called the TRAP MODEL.