Infinite densities for strong anomalous diffusion

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Outline

- Infinite ergodic theory.
- Strong anomalous diffusion.
- Lévy walk model.
- Non-normalizable infinite densities.
Infinite ergodic theory

- Dynamical systems whose invariant measure is infinite.
- Transformations $x_{t+1} = M(x_t)$ with $0 < x_t < 1$.
- For example the Pomeau-Manneville map:

  $$x_{t+1} = x_t + (x_t)^{1+1/\alpha} \mod 1, \quad 0 < \alpha < 1$$

- Gives $x_0, x_1, \ldots x_t, \ldots$ deterministically.
- Exhibits, power law waiting times, $1/f$ noise
Numerical estimate of infinite invariant densities: application to Pesin-type identity

Figure 1. $t_1 \to \zeta(x, t)$ (see equation (6)) for the PM map with $z = 3(\beta = 0.5)$ and $a = 1$. In simulations the times are $t = 10^3, 10^4, 10^5$ (in the figure from bottom to top). Different initial conditions are used to illustrate that the infinite density is not sensitive to the choice of initial conditions: solid line $\zeta(x, 0) = 1$ for $x \in (0, 1)$, circles $\zeta(x, 0) = 2$ for $x \in (0.5, 1)$, squares $\zeta(x, 0) = 2$ for $x \in (0, 1)$. In the limit $t \to 1$ the system approaches the infinite invariant density: the dashed line $\zeta(x) = 0.45x^{1/\beta}$, which is in good agreement with (8) and (9) without any fitting. As follows from equation (7), equation (8) works for $x < x_c$, where $x_c$ is the crossover. For $x \approx x_c$ the finite time $\zeta(x)$ is correctly described by the second line of equation (7) (horizontal dotted lines with no fitting). As $t \to 1$, $x_c = \beta t \to 0$ and since $\beta = 1/2$ we have $\zeta(x) / x^{2/\beta}$ when $x \to 0$, which means the system approaches a non-normalizable state.

Theorem [14] shows that (9) is valid for a large class of maps with a single unstable fixed point on the origin, and which behave like $M(x_t) \sim x_t + axz_t$ for $x_t \to 0$.

Figure 1 demonstrates that when $t \to 1$ equations (8) and (9) describe the infinite invariant density for the PM map. For finite time $t$ and small $x < x_c$ we see deviations in agreement with (7). Since our theory works for small $x$, not surprisingly (8) does not work perfectly for $x > 1$, though deviations seem small to the naked eye.

For the map (3) Thaler has found an exact analytical expression for its infinite invariant density [14]:

$$\zeta(x_t) = B_x \frac{1}{x^{1/\beta}} + (1 + x)^{1/\beta}.$$  (10)

Hence, unlike the PM map where we do not have an exact expression for the infinite density, for the map (3) we can compare simulations with theory in the regime $0 < x < 1$.

Since, as we mentioned for $x_t \to 0$ this map has the same behavior as the PM map, the constant $B_x$ is given by (9). Note that the multiplicative constant $B_x$ is related to our working definition (6) (see further discussion below). In figure 2 we see that $t_1 \to \zeta(x, t)$ slowly converges towards the theoretical infinite density, besides the mentioned deviations close to $x \to 0$. As we increase measurement time the domain $x < x_c$, where deviations from asymptotic theory are observed to be diminishing. In figure 3 we plot $t_1 \to \zeta(x, t)$...
Ergodic theory

Time and ensemble averages are identical

$$\lim_{N \to \infty} \sum_{t=0}^{N} \frac{\mathcal{O}[x_t]}{N} \to \int_{0}^{1} \mathcal{O}(x) \rho(x) dx.$$
Infinite Ergodic theory

- Non-normalized invariant density, \( \rho^{inf}(x) \sim x^{-1/\alpha} \) with \( 0 < \alpha < 1 \)

\[
\int_{0}^{1} \rho^{inf}(x)\,dx = \infty.
\]

- Observable integrable with respect to the infinite density

\[
\lim_{N \to \infty} \left\langle \alpha \sum_{t=0}^{N} \frac{\mathcal{O}[x_t]}{N^\alpha} \right\rangle \to \int_{0}^{1} \mathcal{O}(x)\rho^{inf}(x)\,dx.
\]

- Application of infinite density concept in physics?
Problem of transport in billiards with infinite horizon

M. Courbage,¹ M. Edelman,² S. M. Saberi Fathi,¹ and G. M. Zaslavsky²,³

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Strong Anomalous Diffusion

\[ \langle |x(t)|^q \rangle \sim t^{q \nu(q)}, \quad \nu(q) \neq \text{const} \]

- Brownian motion $\nu(q) = 1/2$.
- Mono-scaling theories are not sufficient or invalid

\[ P(x, t) \neq t^{-\nu} f(x/t^\nu). \]
On *strong* anomalous diffusion

P. Castiglione\(^a,\ast\), A. Mazzino\(^b\), P. Muratore-Ginanneschi\(^c\), A. Vulpiani\(^a\)

\[ \propto q - \text{const} \]

\[ \propto q \]
Bi-Linear spectrum, Physical Examples

\[ q\nu(q) \sim \begin{cases} 
  c_1q & q < q_c \\
  c_2q - c_3 & q > q_c 
\end{cases} \]

- Transport in two dimensional incompressible velocity fields (Vulpiani).
- Deterministic transport in intermittent maps (Artuso and Cristadoro).
- Lorentz gas with infinite horizon (AC, Ott, Zaslavsky).
- Diffusion of cold atoms in optical lattices (Barkai, Lutz)
- Active transport in living cells (Weihs)
- Lévy walks a stochastic framework.

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Experimental evidence of strong anomalous diffusion in living cells

Naama Gal and Daphne Weihs

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Questions

- Is dual scaling an asymptotic property?
- Can we describe the diffusive/ballistic packet?
- Go beyond central limit theorem?
- Introduce an infinite density.
Levy walks

- Zarburdaev, Denisov, Klafter, RMP (2015)
Model

- Pairs of IID RV \((\tau_i, v_i)\).
- PDFs \(\psi(\tau)\) and \(F(v)\).

\[
\begin{align*}
t &= \sum_{i=1}^{N} \tau_i + \tau^* \\
x &= \sum_{i=1}^{N} \chi_i + \chi^* \\
\chi_i &= v_i \tau_i
\end{align*}
\]
Focus of this talk

- Moments of $F(v)$ are finite and $F(v) = F(-v)$.

$$\psi(\tau) \sim \frac{A\tau^{-(1+\alpha)}}{|\Gamma(-\alpha)|} \quad 1 < \alpha < 2$$

- For Lorentz gas $\psi(\tau) \sim \tau^{-3}$.

- $\langle \tau \rangle$ finite, variance of the waiting time diverges.

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Central Limit theorem arguments

- $N \approx t/\langle \tau \rangle$ problem deals with summation of IID RVs?

\[ x \approx \sum_{i=1}^{N} X_i \]

- For $1 < \alpha < 2$ apply Lévy’s central limit theorem

\[ X_i = \tau_i v_i. \]

- However competition between Lévy’s behavior and ballistic tail, makes the problem interesting.

- Lévy’s CLT gives $\langle x^2 \rangle = \infty$, unphysical!
Plan

- Obtain exact expressions for moments $\langle x^n(t) \rangle$
- Use Montroll-Weiss equation and the Faa di Bruno formula.

Moment generating function (Fourier transform)

$$P(k, t) = 1 + \sum_{n=1}^{\infty} \frac{(ik)^n \langle x^n(t) \rangle}{n!}$$

- Sum the infinite series.
- Take the inverse Fourier transform.
- Get the long time limit of $P(x, t)$? **NAIVE.**

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• For two state model \( F(v) = [\delta(v - v_0) + \delta(v + v_0)]/2 \)

\[
\langle x^n(t) \rangle \sim \frac{n}{(n-\alpha)(n+1-\alpha)|\Gamma(1-\alpha)|\langle \tau \rangle} A (v_0)^n t^{n+1-\alpha}
\]

• Summing the series

\[
P_A(k, t) \sim 1 + t^{1-\alpha} \frac{A}{|\Gamma(1-\alpha)|\langle \tau \rangle}\tilde{f}_\alpha(ikv_0 t),
\]

\[
y^2 \left[ \frac{1}{3-\alpha} \ 1F_2 \left( \frac{3-\alpha}{2}; \frac{3}{2}, \frac{5-\alpha}{2}; \frac{-y^2}{4} \right) - \frac{1}{2-\alpha} \ 1F_2 \left( 1 - \frac{\alpha}{2}; \frac{3}{2}, 2 - \frac{\alpha}{2}; \frac{-y^2}{4} \right) \right]
\]
• Take the inverse Fourier transform

\[
P_A(x, t) = \frac{\tilde{A}}{t^\alpha} \left| \frac{x}{v_0 t} \right|^{-(1+\alpha)} \left[ 1 - \left| \frac{\alpha-1}{\alpha} \frac{x}{v_0 t} \right| \right] \quad \text{for} \quad 0 \neq |x| < v_0 t
\]

• Non-normalizable density. TRASH SOLUTION?

• Ballistic \( x/t \) scaling.

\[
\tilde{A} = A\alpha / 2v_0 \langle \tau \rangle |\Gamma(1 - \alpha)|
\]
What do simulations say?

\[ \alpha P(x,t) \]

- infinite covariant density

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The Infinite covariant density (ICD)

\[
\lim_{t \to \infty} t^\alpha P(x, t) = I_{cd}(\bar{v}) \quad \bar{v} \equiv x/t
\]

For example

\[
I_{cd}(\bar{v}) = K_\alpha c_\alpha |\bar{v}|^{-(1+\alpha)} \left[ 1 - \frac{\alpha-1}{\alpha} \frac{v_0}{|\bar{v}|} \right]
\]

Two types of observables: 
integrable \((\bar{v}^2)\) and non-integrable \((\bar{v}^0)\) 
with respect to the ICD.

\[
K_\alpha = A \langle |v|^\alpha \rangle \cos(\pi\alpha/2) / \langle \tau \rangle \quad c_\alpha = \sin(\pi\alpha/2) \Gamma(1 + \alpha) / \pi.
\]
Lévy’s central limit theorem implies that for the center of the packet

\[ \frac{\partial P_{cen}(x,t)}{\partial t} = K_\alpha \nabla^{\alpha} P_{cen}(x, t) \]

- $K_\alpha$ the anomalous diffusion coefficient can be used to estimate the ICD.

- Observable integrable with respect to Lévy’s PDF, i.e., $|x|^q$ and $0 < q < \alpha$, is non integrable with respect to the ICD.
ICD is complementary to the central limit theorem

\[ I_{cd}(\bar{v}) \sim K_\alpha c_\alpha |\bar{v}|^{-(1+\alpha)} \text{ for } \bar{v} \to 0. \]

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\[ \langle |x(t)|^q \rangle = \begin{cases} \frac{M_q^<}{q} t^{q/\alpha} & q < \alpha, \\ M_q^> t^{q+1-\alpha} & q > \alpha. \end{cases} \]
Relation between the ICD and velocity distribution $F(v)$

$$\mathcal{I}_{CD}(\bar{v}) = B \left[ \frac{\alpha \mathcal{F}_\alpha(|\bar{v}|)}{|\bar{v}|^{1+\alpha}} - \frac{\alpha-1}{|\bar{v}|^\alpha} \mathcal{F}_{\alpha-1}(|\bar{v}|) \right]$$

where

$$\mathcal{F}_\alpha(\bar{v}) = \int_{|\bar{v}|}^{\infty} dv\, v^\alpha F(v)$$

$$B = \bar{c}_\alpha K_\alpha \langle |v|^{\alpha} \rangle .$$

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Summary

- Dual scaling implies active transport is both quasi ballistic and super diffusive.
- Infinite density is complementary to the central limit theorem.
- Two classes of observables integrable and non-integrable with respect to infinite covariant density.
- Infinite densities describe statistics of a growing number of physical models.
Thanks and Ref.

- Further examples: see Johannes Schulz (talk) and Erez Aghion (poster).
Things to do

- In experiment moment $\langle |x(t)|^q \rangle$ has different time regimes, ballistic then diffusive etc, so emphasize that $q\nu(q)$ is found in long time limit (or show the moments as function of time $t$).

- Define $P(x, t)$.

- Take fig. from Denisov of trajectory of particle in infinite Lorentz gas. (see above needss to be fixed).

- Emphasize that we take $t \to \infty$ first (in calculation of moments) then obtain infinite series which is summed and inverse Fouriered. Limits do not commute.

- Use fig. from PRE (not only $x > 0$).