Control and Manipulation
of the Kerr-lens Mode-locking
Dynamics in Lasers

Shai Yefet

Department of Physics

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This work was carried out under the supervision of

Dr. Avi Pe’er

Deptartment of Physics

Bar-Ilan University
This work is dedicated to my family,

my wife *Keren* and my *parents* for their love and support all these years.
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Abstract

This thesis describes theoretical and experimental studies of mode-locked solid-state lasers, specifically a Kerr-lens mode-locked titanium:sapphire laser. This laser is the major source for ultrashort optical pulses, which provides an extremely important tool to investigate ultrafast processes in physics, chemistry and biology. In this work, we investigate the underlying physics of these lasers and manipulate it in order to improve the laser performance. By presenting novel cavity designs and configurations, we overcome many of the disadvantages and limitations of standard designed Ti:sapphire optical cavities. The study is divided into 3 stages, each submitted for publication as a separate paper:

• In the first stage, we demonstrate precise control over the mode competition in a mode locked Ti:sapphire oscillator by manipulation and spectral shaping of the gain properties, thus steering the mode competition towards a desired, otherwise inaccessible, oscillation.

• In the second stage, we demonstrate a Kerr-lens mode locked folded cavity using a planar (non-Brewster) Ti:sapphire crystal as a gain and Kerr medium, thus cancelling the nonlinear astigmatism caused by a Brewster cut Kerr medium, which otherwise needs a special power spe-
cific compensation.

- In the third stage, we explore experimentally a new regime of operation for mode locking in a Ti:Sapphire laser with enhanced Kerr nonlinearity, where the threshold for pulsed operation is lowered below the threshold for continuous-wave operation. In this regime, pulsed oscillation can be realized directly from zero continuous-wave oscillation.
Chapter 1

Introduction

The interest in optical systems was intensified since the laser invention \([1]\) and the realization of nanosecond pulses \(10^{-9}\text{s}\) \([2]\). During the following years, the interest and research in the field of ultrafast optics grew rapidly and it is still the subject of active research. In the past decades, progress in this field has been growing rapidly, leading to the realization of optical sources which produce pulses as short as a few femtoseconds \(10^{-15}\text{s}\) \([3, 4, 5, 6]\).

1.1 Motivation

Many scientific and industrial areas incorporate ultrashort pulses since they hold great value in their temporal and spectral domains\(^1\). These pulses enable the investigation of ultrafast light-matter interactions \([7]\) and other ultrafast processes in chemistry \([8]\) and biology \([9]\). A comprehensive review regarding the applications of ultrashort laser systems can be found in \([10, 11]\). Therefore, the realization of sources producing ultrashort pulses is highly attractive.

\(^1\)These properties will be discussed in detail in Ch. 3.2.2
Chapter 1. Introduction

One of the most popular sources of ultrashort pulse generation is the Kerr-lens mode-locked (KLM) Titanium:Sapphire (TiS) laser [12], common in almost every research laboratory engaged with ultrafast phenomena or high resolution spectroscopy. Even though these lasers are common and work well, standard designed oscillators still suffer from several inherent disadvantages and limitations, such as:

- Limited control of the oscillation spectrum.
- Non-linear optical aberrations that cannot be compensated simply.
- Deficient exploitation of the nonlinear mode locking mechanism.

Finding solutions to this problems holds great value and benefit, and stands at the center of this thesis.

1.2 Overview

Since this work deals specifically with TiS mode-locked lasers, a detailed description of this laser is given in the following chapters, as follows:

- Chapter 2 provides a concise review of the basic principles of general laser operation.
- Chapter 3 details the basic concepts of various types of pulse generation mechanisms.
- Chapter 4 describes the dynamics and the master equation of pulse propagation.
1.2. Overview

- Chapter 5 describes various types of mode locking techniques.
- Chapter 6 describes the basic design of a KLM TiS cavity.
- Chapter 7 describes the ABCD matrix technique for cavity analysis and the KLM TiS cavity is analysed in Ch.8.
- Chapters 9 and 10 focuses on dispersion management and additional astigmatism compensation, respectively.
- Chapter 11 provides a qualitative review of the experimental behaviour of the KLM TiS cavity. A brief review regarding optimization and improvement of laser parameters using nonstandard cavity designs is also provided.

The main research of this work is gathered in the last three chapters. The research addresses the above mentioned problems and provides simple solutions in terms of non-standard cavity designs:

- Chapter 12 presents an intracavity shaper, allowing flexible and power efficient control on the spectral gain.
- Chapter 13 presents a novel type of cavity folding, eliminating from the source the nonlinear aberrations.
- Chapter 14 presents a cavity with enhanced nonlinearity and ultra-low threshold, which pushes the mode locking mechanism to a new regime where the threshold for mode locking can be lower than the threshold for continuous-wave operation.
Chapter 2

Basic principles of laser operation

2.1 Light amplification

The stimulated emission process is the core principal of laser operation. In this mechanism, illustrated in Fig.2.1(a), an exited atom interacts with a photon that causes the atom to de-excite while emitting a second photon which is coherent with the first one. This effect can be amplified in an optical cavity, as illustrated in Fig.2.1(b), which must include three essential elements:

- **Active medium**, which contains the atoms used for stimulated emission.

- **Pumping mechanism**, which delivers energy into the active medium to excite the atoms.
2.1. Light amplification

- Optical feedback elements, which enables the amplification of the photons through circulation.

As the photons circulate through the cavity, they are being amplified by the active medium through stimulated emission while experiencing certain amount of losses. Laser operation is possible only when the gain exceeds the cavity losses. When this threshold is crossed, laser power builds up quickly and eventually decreases the gain as it consumes more and more of the finite amount of the pumping energy. Finally, the laser reaches a steady state operation where the gain and the losses equalize. For a comprehensive and detailed study of laser operation readers are referred to the classic literature [13, 14].
Chapter 2. Basic principles of laser operation

2.2 Continuous-wave operation

The optical cavity can support different longitudinal modes that satisfy its boundary conditions. As illustrated in Fig. 2.2(a), the cavity supports modes that satisfy $L = m\lambda/2$, where $L$ is the cavity length, $\lambda$ is the mode wavelength and $m$ is an integer number. Due to the active medium, the modes that can physically oscillate in the cavity are only the modes that match the active medium emission spectrum, as illustrated in Fig. 2.2(b).

When a laser operates in a continuous-wave (CW) operation, one or more longitudinal modes can oscillate in the cavity simultaneously. This state of multi-mode laser operation is characterized by the fact that each oscillating
2.2. Continuous-wave operation

mode has a random phase. Therefore, the modes are randomly interfering with one another and the laser output will be noise-like intensity as a function of time, periodic in the cavity round trip time. Typically however, the number of oscillating modes will be reduced drastically due to mode competition. Since all the modes are competing for a finite amount of pump power and the active medium provides non-equal gain to the participating modes during the lasing process, the modes with the higher gain quickly overtake the oscillator at the expense of modes with lower gain. Therefore, in a mode competition situation, multi-mode operation is suppressed into single-mode operation, which is characterized by a clean sinusoidal wave at the laser output with an extremely narrow-band spectrum, ideally limited by the Schawlow-Townes equation [15].

In reality, even with mode competition, a pure single-mode operation requires some effort to achieve, especially when the separation between adjacent longitudinal modes is much smaller than the bandwidth of the active medium emission spectrum. In this case, there are many modes for which the gain is maximal and identical and the laser can randomly jump between different modes with equal gain. For example: a typical TiS cavity has $\approx 10^6$ modes, while an Argon or HeNe laser has only a few.
Chapter 3

Pulsed operation

For a resonator to produce light pulses, two major methods are common:

- Q-switching
- Mode locking

3.1 Q-switching

In Q-switching \cite{16}, the electromagnetic energy is stored inside the gain medium until it is released at once, due to a quick change of the resonator $Q$-factor. Using Q-switching, pulses with duration typically in the nanosecond regime can be achieved. Mode locking on the other hand, can generate far shorter pulses, as explained hereon. The realization of such ultrashort pulses, requires an active gain medium with broad enough spectral bandwidth, hence the TiS crystal is extremely useful for ultrashort pulse generation due to its broad emission spectrum and its excellent mechanical properties\footnote{1These properties will be discussed in Ch. 6.1}.
3.2 Mode locking

3.2.1 Basic concept

A laser is said to be mode-locked (ML) if many longitudinal modes are oscillating together with well defined and fixed phase relation between them, as opposed to CW operation [17]. As seen in Fig.3.1 this phase relation causes the modes to constructively interfere with one another only in a short period of time to form a pulse with higher intensity, while destructively interfering at all other times.

In other words, the mode locking mechanism squeezes the electro-magnetic energy, which would otherwise spread over a long time in CW operation, into an extremely short period of time with high peak power. Eventually, this mechanism will reach a steady state in which the pulse circulates through the resonator maintains its temporal shape. Pulse formation can also be qualitatively described based on the Kuizenga-Siegman theory [18], describing the interplay between two mechanisms. The first mechanism is due to some nonlinear effect which acts as a pulse shortening mechanism. This effect monotonically decreases in magnitude as the pulse becomes shorter and shorter. The second mechanism is the finite gain bandwidth of the active medium which acts as a pulse broadening mechanism. This effect is monotonically increases as the pulse becomes shorter and shorter. When these two mechanisms are in balance, the pulse reaches a steady state.
Figure 3.1: Pulse formation by a fixed phase relation between many longitudinal modes.
3.2. Mode locking

The relation between the temporal duration of the pulse and its spectral bandwidth can be obtained from the time-energy uncertainty principle is given by:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$  \hspace{1cm} (3.1)

Since the energy of a photon is $E = h\omega$, Eq.3.1 can be rewritten as:

$$\Delta \omega \Delta t \geq \frac{1}{2}$$  \hspace{1cm} (3.2)

From Eq.3.2 it follows that in order for photons to be localized in time, they must have many frequency components. The temporal and frequency standard deviations $\Delta \omega$ and $\Delta t$ can be calculated by the general formula:

$$\Delta \omega = \sqrt{\langle \omega^2 \rangle - \langle \omega \rangle^2}$$  \hspace{1cm} (3.3)

where $\langle \omega^2 \rangle$ and $\langle \omega \rangle$ can be calculated using

$$\langle f(\omega) \rangle = \frac{\langle \Psi | f(\omega) | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\int_{-\infty}^{\infty} \Psi^* f(\omega) \Psi d\omega}{\int_{-\infty}^{\infty} |\Psi|^2 d\omega}$$  \hspace{1cm} (3.4)

We represent the pulse $E(t)$ as a carrier wave with frequency $\omega_0$, oscillating within a Gaussian envelope:

$$E(t) = E_0 \exp \left( -\frac{t^2}{2\sigma_t^2} \right) \exp(-i\omega_0 t)$$  \hspace{1cm} (3.5)

\[^2\text{The same formulas can be used to calculate} \langle t^2 \rangle \text{ and} \langle t \rangle.\]
Chapter 3. Pulsed operation

The Fourier transform of Eq. 3.5 is also a Gaussian, centered at \( \omega_0 \), given by:

\[
\tilde{E}(\omega) = \tilde{E_0} \exp\left(-\frac{(\omega - \omega_0)^2}{2\sigma_{\omega}^2}\right)
\]  

(3.6)

Substituting Eq. 3.6 into Eq. 3.4 (using \( \Psi = \tilde{E}(\omega) \)), one can calculate \( \langle \omega^2 \rangle \) (using \( f(\omega) = \omega^2 \)) and \( \langle \omega \rangle \) (using \( f(\omega) = \omega \)), resulting in:

\[
\langle \omega^2 \rangle = \omega_0^2 + \frac{\sigma_{\omega}^2}{2} \\
\langle \omega \rangle = \omega_0
\]

(3.7)

Hence, from Eq. 3.3 it follows that the standard deviation in the frequency domain is \( \Delta \omega = \sigma_{\omega}/\sqrt{2} \), and the same can be done for the temporal domain by using Eq. 3.5 resulting in \( \Delta t = \sigma_t/\sqrt{2} \).

Therefore, Eq. 3.2 becomes:

\[
\Delta \omega \Delta t = \frac{\sigma_{\omega} \sigma_t}{2} \geq \frac{1}{2} \quad \implies \quad \sigma_{\omega} \sigma_t \geq 1
\]

(3.8)

where unity is the minimum time-bandwidth product for a Gaussian shaped pulse and a pulse that satisfies \( \sigma_{\omega} \sigma_t = 1 \) is said to be transform limited.

### 3.2.2 Time/frequency domain

As the pulse circulates through the cavity, it is coupled out from the laser each time it hits the output coupler. The output beam therefore forms a pulse train. In the time domain, illustrated in Fig. 3.2(a), the source emits extremely short packets of electro-magnetic radiation with temporal duration of \( \sigma_t \), separated by intervals of the round trip time in the cavity \( \tau_r \).
3.2. Mode locking

Mathematically, this pulse train $E(t)$ can be represented as a convolution of a single pulse with a sum of equidistant delta functions:

$$E(t) = E_0 \left[ \sum_m \delta(t - m\tau_r) \right] \ast \left[ \exp \left( -\frac{t^2}{2\sigma_t^2} \right) \exp(-i\omega_0 t) \right],$$  \hspace{1cm} (3.9)

where the pulse was assumed to have a Gaussian shape, $\omega_0$ is the optical carrier frequency and $m$ is an integer. Using the convolution theorem, the Fourier transform of Eq.3.9 is:

$$\tilde{E}(\omega) = \tilde{E}_0 \left[ \sum_m \delta(\omega - m\omega_{rep}) \right] \cdot \left[ \exp \left( -\frac{(\omega - \omega_0)^2}{2\sigma_\omega^2} \right) \right]$$  \hspace{1cm} (3.10)

where $\omega_{rep} = 2\pi/\tau_r$. When all the frequency components in the spectrum have the same phase, the pulse is the shortest possible with the given spectrum, and hence termed transform limited\(^3\) ($\sigma_t\sigma_\omega = 1$). Therefore, in the frequency domain illustrated in Fig.3.2(b), the field contains a large number of discrete, equidistant and phase-related frequencies contained within a broad envelope, which is known as a frequency comb.

3.2.3 Pulse as a frequency comb

The spectral representation of a pulse train in Eq.3.10 and Fig.3.2(b) illustrates the term "frequency comb"\(^4\). Frequency combs is an extremely important application of ultrashort pulses that received great attention since the Nobel Prize in Physics was awarded to J. L. Hall and T. W. Hänsch \[19\] \[20\] \[21\]. Equation 3.10 however, assumes that the zero frequency tooth, for which $m = 0$, is positioned at $f_0 = 0$. In general this is not the case and the comb frequencies can be expressed as: $f_m = mf_{rep} + f_0$, where $f_{rep} = 1/\tau_r$

\(^3\)The reason for a pulse to be a non-transform limited, will be explained in Ch.9.1

\(^4\)Using the simple relation $\omega = 2\pi f$
and \( f_0 \) is an offset frequency, not necessarily zero.

The origin of the offset frequency is from the carrier-envelope offset phase, \( \Delta \phi_{ce} \), which originates from the difference between the group velocity and the phase velocity in dispersive materials, as follows:

The time taken for the carrier traveling at the phase velocity, to complete a single round trip is: \( \tau_p = L/v_p \) where \( L \) is the cavity length. Equivalently, the time taken to the envelope traveling at the group velocity, to complete a single round trip is: \( \tau_r = L/v_g = 1/f_{rep} \). The phase velocity is given by: \( v_p = c/n \), and the group velocity is given by: \( v_g = c/(n + \omega_0 \frac{dn}{d\omega}) \), where the refractive index \( n \) and its derivative are calculated at the carrier frequency \( \omega_0 \). Hence, the relative phase between the carrier and the envelope after one period can be expressed by:

\[
\Delta \phi_{ce} = \omega_0 (\tau_r - \tau_p) = \omega_0 L \left( \frac{1}{v_g} - \frac{1}{v_p} \right) = \frac{L \omega_0^2}{c} \frac{dn}{d\omega} \tag{3.11}
\]

During one pulse period, the phase accumulated by a single comb component is:

\[
\phi_m = \omega_m \tau_r = 2\pi f_m/f_{rep} \tag{3.12}
\]

But the phase \( \phi_m \) accumulated by a single comb component is also the sum:

\[
\phi_m = 2\pi m + \Delta \phi_{ce} \tag{3.13}
\]

Equating Eq.\(3.12\) and Eq.\(3.13\) with a simple rearrangement, yields:

\[
f_m = mf_{rep} + f_0 \quad \quad f_0 = f_{rep}(\Delta \phi_{ce}/2\pi) \tag{3.14}
\]
Figure 3.2: Pulse train in (a) temporal and (b) spectral domain.
Chapter 4

Pulse propagation equation

In the following, we derive the differential equation that describes the propagation of pulses inside a dispersive, nonlinear medium.

4.1 The nonlinear Schrödinger equation

The wave equation is given by:

\[ \nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P}{\partial t^2} \]  

(4.1)

where \( P \) is the polarization vector, which can be related to \( E \) by:

\[ P_i = \epsilon_0 \left( \chi^{(1)} \cdot E + \chi^{(2)} : EE + \chi^{(3)} : EEE + \ldots \right) \]

(4.2)

\[ P_i = \epsilon_0 \left[ \sum_j \chi^{(1)}_{ij} E_j + \sum_{jk} \chi^{(2)}_{ijk} E_j E_k + \sum_{jkl} \chi^{(2)}_{ijkl} E_j E_k E_l + \ldots \right] \]

where \( \chi^{(1)}, \chi^{(2)}, \ldots \) are the first, second and higher order susceptibility tensors.
4.1. The nonlinear Schrödinger equation

To describe a propagating pulse, the electric field $E(r, t)$ can be presented as:

$$E(r, t) = \frac{1}{2} \hat{x} \left( F(x, y) A(z, t) \exp[i k_0 z - i \omega_0 t] + c.c. \right) \quad (4.3)$$

In Eq.4.3, the field is polarized in the $\hat{x}$ direction and propagates along the $z$ direction at the phase velocity:

$$v_p = \omega_0 / k_0 = \omega_0 \frac{c}{n(\omega_0) \omega_0} = \frac{c}{n(\omega_0)} \quad (4.4)$$

In addition, $F(x, y)$ is the spatial distribution of the field and $A(z, t)$ represents the envelope in which the carrier wave oscillates at the central frequency $\omega_0$.

When a pulse propagates through dispersive media, it is subjected to dispersion. To include dispersion effects, the frequency dependent wavenumber $k(\omega)$ is expanded in a Taylor series around the pulse central frequency $\omega_0$:

$$k(\omega) = n(\omega) \frac{\omega}{c} = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} (\omega - \omega_0)^m$$

$$= \beta_0 + \beta_1 (\omega - \omega_0) + \frac{\beta_2}{2} (\omega - \omega_0)^2 + ... \quad (4.5)$$

where

$$\beta_m = \left( \frac{d^m k}{d\omega^m} \right) \omega_0 = \frac{1}{c} \left( \frac{m}{m-1} n + \omega_0 \frac{d^n}{d\omega^m} \right) \quad (m = 0, 1, 2, ...) \quad (4.6)$$

We calculate the first four parameters of $\beta_m$ which will be relevant to this work (all the derivatives are calculated at the central frequency $\omega_0$):
\( \beta_0 = \frac{\omega_0 n(\omega_0)}{c} \)

\( \beta_1 = \frac{1}{c} \left( n + \omega_0 \frac{dn}{d\omega} \right) \)

\( \beta_2 = \frac{1}{c} \left( 2 \frac{dn}{d\omega} + \omega_0 \frac{d^2 n}{d\omega^2} \right) \)

\( \beta_3 = \frac{1}{c} \left( 3 \frac{d^2 n}{d\omega^2} + \omega_0 \frac{d^3 n}{d\omega^3} \right) \) \tag{4.7}

Here, the first term in the series \( \beta_0 \) is simply the wavenumber \( k_0 \) corresponding to the central frequency \( \omega_0 \). The second term \( \beta_1 \) is related to the group velocity in which the pulse envelope \( A(z,t) \) travels, through:

\[ v_g = \frac{d\omega}{dk} = \frac{1}{\beta_1}. \]

The third and higher terms are responsible for pulse broadening: \( \beta_2 \) is called the group velocity dispersion (GVD) and \( \beta_3 \) is called the third order dispersion (TOD).

It was shown [22], that Eq.4.1 can be reduced, after some assumptions and approximations, to a differential equation for the envelope \( A(z,T) \):

\[ \frac{\partial A}{\partial z} + \frac{\alpha}{2} A + \frac{i \beta_2}{2} \frac{\partial^2 A}{\partial T^2} - \frac{\beta_3}{6} \frac{\partial^3 A}{\partial T^3} - i \gamma |A|^2 A = 0 \] \tag{4.8}

where the time variable \( t \) is replaced with a temporal frame of reference that moves at the group velocity, defined by: \( T = t - z/v_g \). The first term in Eq.4.8 is the rate at which the envelope of the pulse changes as a function of \( z \). The second term represents the absorption, where \( \alpha \) is the absorption coefficient of the intensity which is proportional to \( |E|^2 \). Since the intensity decreases as \( \sim \exp[-\alpha z] \), the envelope \( A(z,T) \) will decrease as \( \sim \exp[-(\alpha/2)z] \). The third and forth terms are the quadratic and cubic terms in Eq.4.5 both represent the dispersion effect. The fifth term in Eq.4.8 represents the self
phase modulation (SPM) effect governed by the parameter $\gamma$ given by:

$$\gamma = \frac{n_2\omega_0}{cS} \quad (4.9)$$

where $S$ is the effective area of the mode, defined by:

$$S = \frac{\left(\int_{-\infty}^{\infty} \left| F(x, y) \right|^2 dx dy \right)^2}{\int_{-\infty}^{\infty} \left| F(x, y) \right|^4 dx dy} \quad (4.10)$$

Substituting a Gaussian profile $F(x, y) \sim \exp\left[-\left(x^2/\omega_x^2\right) - \left(y^2/\omega_y^2\right)\right]$ into Eq.4.10 yields: $S = \pi \omega_x \omega_y$. Since Eq.4.8 is similar to the Schrödinger equation including a nonlinear potential, it is referred to as: the nonlinear Schrödinger (NLS) equation.

It is clear from Eq.4.8 that by eliminating all terms except the first two and solving $\partial A/\partial z = -(\alpha/2)A$, it follows that the amplitude of $A(z, T)$ decreases exponentially with $z$. Hence, the envelope $A(z, T)$ can be defined as:

$$A(z, T) = \sqrt{P_0} \exp\left[-(\alpha/2)z\right] U(z, T) \quad (4.11)$$

where $P_0$ is the pulse peak power and $U(z, T)$ is a dimensionless envelope.

Hence, the NLS equation can be presented by:

$$\frac{\partial U}{\partial z} + \frac{i\beta_2}{2} \frac{\partial^2 U}{\partial T^2} - \frac{\beta_3}{6} \frac{\partial^3 U}{\partial T^3} - i\gamma P_0 \exp(-\alpha z) |U|^2 U = 0 \quad (4.12)$$

### 4.2 Dispersion

Let us consider the solution to Eq.4.12 including only dispersion effects. By eliminating the term proportional to $|U|^2U$, Eq.4.12 is reduced to:

$$\frac{\partial U}{\partial z} + \frac{i\beta_2}{2} \frac{\partial^2 U}{\partial T^2} - \frac{\beta_3}{6} \frac{\partial^3 U}{\partial T^3} = 0 \quad (4.13)$$

1The origins of SPM will be discussed in Ch.5.2.2
Chapter 4. Pulse propagation equation

This equation can be solved using the Fourier transform. We can define $U(z, T)$ as the inverse Fourier transform of $\tilde{U}(z, \omega)$:

$$U(z, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(z, \omega) \exp(-i\omega T) d\omega \quad (4.14)$$

By substituting Eq. 4.14 in Eq. 4.13, one can write Eq. 4.13 in the frequency domain:

$$\frac{\partial \tilde{U}}{\partial z} = i \left( \frac{\beta_2}{2} \omega^2 + \frac{\beta_3}{6} \omega^3 \right) \tilde{U} \quad (4.15)$$

The solution to Eq. 4.15 is given by:

$$\tilde{U} = \tilde{U}_0 \exp \left[ i z \left( \frac{\beta_2}{2} \omega^2 + \frac{\beta_3}{6} \omega^3 \right) \equiv \varphi(\omega) \right] \quad (4.16)$$

where $\tilde{U}_0$ is the Fourier transform of $U(z, T)$ at $z = 0$:

$$\tilde{U}_0 = \int_{-\infty}^{\infty} U(0, T) \exp(i\omega T) dT \quad (4.17)$$

Finally the solution to $U(z, T)$ can be presented by:

$$U(z, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}_0 \exp[i\varphi(\omega) - i\omega T] d\omega \quad (4.18)$$

where $z = 0$ is where dispersion starts.

4.3 Self phase modulation

Let us consider the solution to Eq. 4.12 including only the self phase modulation effect. By setting $\beta_2 = \beta_3 = 0$, Eq. 4.12 is reduced to:

$$\frac{\partial U}{\partial z} - i\gamma P_0 \exp(-\alpha z) |U|^2 U = 0 \quad (4.19)$$
Equation 4.19 can be solved by substituting into the equation a general representation of $U$ as a complex number with amplitude and additional nonlinear phase: $U = a \cdot \exp(i\phi_{NL})$. After separating into real and imaginary parts, we obtain:

$$\frac{\partial a}{\partial z} = 0 \quad \frac{\partial \phi_{NL}}{\partial z} = \gamma P_0 |a|^2 \exp(-\alpha z) \quad (4.20)$$

From Eq. 4.20 it follows that $a$ is independent of $z$. If SPM starts at $z = 0$ then the nonlinear phase $\phi_{NL}$ should be zero at $z = 0$. Hence, we obtain: $U(0, T) = a$. It also follows from Eq. 4.20 that $\phi_{NL}$ can be obtained by simple integration while demanding $\phi_{NL}(0, T) = 0$:

$$\phi_{NL}(z, T) = \frac{\gamma P_0}{\alpha}|U(0, T)|^2[1 - \exp(-\alpha z)] \quad (4.21)$$

Finally, the solution of Eq. 4.19 is given by:

$$U(z, T) = U(0, T) \cdot \exp[i\phi_{NL}(z, T)] \quad (4.22)$$

### 4.4 Split-step Fourier method

In general, the self phase modulation and dispersion effects operate simultaneously on the pulse as it propagates through the material. Therefore, Eq. 4.12 is solved numerically, using an approximated method called: Split-step Fourier method.

Equation 4.12 can be written in the form:

$$\frac{\partial U}{\partial z} = (\hat{D}1 + \hat{D}2)U \quad (4.23)$$

where the operators $\hat{D}1$ and $\hat{D}2$ are defined by:
\[
\hat{D}_1 = -\frac{i\beta_2}{2} \frac{\partial^2}{\partial T^2} + \frac{\beta_3}{6} \frac{\partial^3}{\partial T^3}
\]
\[
\hat{D}_2 = i\gamma P_0 \exp(-\alpha z)|U|^2
\]

(4.24)

The partial derivative is defined as:

\[
\frac{\partial U}{\partial z} = \left[ U(z+h,T) - U(z,T) \right]/h
\]

at the limit \( h \to 0 \). Using this definition in Eq.4.23 yields:

\[
U(z+h,T) = [1 + h(\hat{D}_1 + \hat{D}_2)]U(z,T).
\]

At the limit \( h \to 0 \) we obtain

\[
1 + h(\hat{D}_1 + \hat{D}_2) = \exp[h(\hat{D}_1 + \hat{D}_2)],
\]

hence finally:

\[
U(z+h,T) = \exp(h\hat{D}_2) \exp(h\hat{D}_1)U(z,T)
\]

(4.25)

Therefore, the NLS equation (Eq.4.12) can be solved by dividing the material to many infinitely thin slices with thickness \( h \), while in each slice the dispersion and SPM are applied separately one after the other. Numerically, the slice thickness \( h \) should be considerably smaller than two characteristic length scales, given by:

\[
L_{\hat{D}_1} = \frac{\sigma_t^2}{\beta_2} \quad L_{\hat{D}_2} = \frac{1}{\gamma P_0}
\]

(4.26)

where \( \sigma_t \) is the pulse temporal duration. These length scales are associated with the operators \( \hat{D}_1 \) and \( \hat{D}_2 \) and provide an estimation to the lengths in which dispersion or SPM becomes important for the evolution of \( U(z,T) \). Therefore, the thickness \( h \) of a given slice must be considerably smaller than both \( L_{\hat{D}_1} \) and \( L_{\hat{D}_2} \) so neither dispersion nor SPM will play a significant role as \( U(z,T) \) propagates through the slice\(^2\). In addition, since both length scales in Eq.4.26 depend on the pulse parameters \( P_0 \) and \( \sigma_t \), the thickness of the slices must be updated numerically as the pulse propagates through the material.

\(^2\)Obviously, from the definition of \( L_{\hat{D}_1} \) it is clear that only GVD is considered and not TOD.
Chapter 5

Mode locking techniques

Mode locking can be achieved by several techniques that can be divided into two categories [18, 23, 24, 25]:

- **Active mode locking**, which requires an external modulation signal inside the cavity to enforce pulsed operation.

- **Passive mode locking**, where preference of pulsed operation is introduced by some additional intra-cavity nonlinear response.

5.1 Active mode locking

Active mode locking can be achieved by intra-cavity amplitude modulation (AM) or frequency modulation (FM) at an exact multiple of the cavity repetition rate \( f_{\text{rep}} \) [18, 23]. The first method commonly uses the acousto-optic effect, in which the modulator acts as a fast shutter in the cavity. Pulsed operation is achieved by synchronizing the modulation rate with the round trip time of the pulse, so that a pulse will always pass through the modulator.
Chapter 5. Mode locking techniques

when the "shutter is open”. The second method uses the electro-optic effect (as in Pockels-cell), in which the modulator introduces a frequency shift to the light passing through it. A pulse will experience zero frequency shift, while a single frequency mode will experience a frequency up-shift (or down-shift) which will eventually sweep the mode out from the active medium gain bandwidth after many round trips.

5.2 Passive mode locking

5.2.1 Saturable absorber

Passive mode locking can be achieved by introducing a saturable absorber into the cavity, which absorbs light with low intensities but transmits light with high intensities due to saturation of the absorbing mechanism [24]. Therefore, CW will suffer constant absorption losses, whereas a pulse with high peak power will saturate the absorber, reducing the attenuation considerably. The saturable absorber then acts as a shutter, but with much faster modulation rate than any electronic based shutter. Since the pulse itself opens the shutter (saturates the absorber), the faster the response time of the saturable absorber to intensity variations, the shorter the temporal duration of the pulse can be [25].

5.2.2 Optical Kerr effect

Passive mode locking can also be achieved by introducing an artificial saturable absorber through the process of the optical Kerr effect [26] [27] [28] [29] [30]. Using the Kerr effect, pulses as short as a few femtoseconds can be
5.2. Passive mode locking

achieved, while most real saturable absorbers produce pulses in the picosecond scale to the long femtosecond scale. For a more detailed and mathematical review regarding the theory of active/passive mode-locking, slow/fast saturable absorbers and Kerr-media, the master equation, pulse dynamics and laser noise can be found in [31, 32, 33].

The Kerr effect is a third-order nonlinear process in which the refractive index of the material is intensity dependent [34], given by:

\[
    n(I) = n_0 + n_2 I
\]

where \( n_0 \) and \( n_2 \) are the linear and nonlinear parts of the refractive index respectively and \( I \) is the light intensity. This effect is practically instantaneous and can support ideally “infinitely” short pulses. In practice, the pulse duration will limited by other considerations, e.g. gain bandwidth, dispersion compensation, etc. Since \( n_2 \) is \( \approx 3 \cdot 10^{-16} \text{cm}^2/\text{W} \) for the Sapphire crystal [35], only high peak power pulses will be sensitive to this effect. Pulsed operation can be obtained from this mechanism by noting that the intensity is both time and lateral space dependent \( I(x, t) \) and has a bell shaped variation in both domains.

**Spatial domain: Kerr-lensing**

In the spatial domain, the center of the beam, where the intensity is higher will experience a larger refractive index compared to the wings, causing the central part to move slower, resulting in a focusing effect of the beam, as seen in Fig.5.1. The analogy to a simple focusing lens is immediate. In a simple lens, the refractive index is constant for both the center and the wings of
Figure 5.1: Schematic diagram of the optical Kerr effect as a self focusing mechanism.

The beam, but the center travels through extra material thickness. In a Kerr medium, the thickness is constant but the refractive index is higher at the center.

The dioptric power of the nonlinear Kerr lens caused by a thin slice of Kerr medium with thickness $z$ and a nonlinear coefficient $n_2$ is given by:

$$\frac{1}{f} = \frac{4n_2 z P}{\pi \omega^4}$$

(5.2)

where $P$ is the pulse peak power and $\omega$ is the mode radius.\textsuperscript{2} This intensity dependent nonlinear lens can now be used as a stabilizing mechanism.

\textsuperscript{1}To be considered as thin, the material thickness should be considerably smaller than the Rayleigh range of the mode.

\textsuperscript{2}Note that Eq. 5.2 holds only for a circular beam where $\omega_x = \omega_y = \omega$. For a non-circular beam with $\omega_x \neq \omega_y$, the dioptric power will be different for each plane and both planes will be coupled to one another. This is explained in detail in Ap. A.
5.2. Passive mode locking

for high intensity pulsed oscillations in an otherwise unstable cavity for CW

Temporal domain: self phase modulation

In the temporal domain, the intensity dependent refractive index modulates the instantaneous frequency of the pulse around the central frequency $\omega_0$, as follows:

The phase that the pulse accumulates through a Kerr medium is:

$$\phi = \omega_0 t - k_0 n(I) L$$
$$= \omega_0 t - \frac{\omega_0}{c} n_0 L - \frac{\omega_0}{c} n_2 I(t) L$$

(5.3)

Substituting a Gaussian intensity into Eq. 5.3 and omitting the constant phase, yields:

$$\phi = \omega_0 t - \frac{\omega_0}{c} n_2 L I_0 \exp \left( -\frac{t^2}{\sigma_t^2} \right)$$

(5.4)

The instantaneous frequency of the pulse is:

$$\omega(t) = \dot{\phi} = \omega_0 + \epsilon t \exp \left( -\frac{t^2}{\sigma_t^2} \right)$$

(5.5)

where $\epsilon = \frac{(2\omega_0 L n_2 I_0)}{(c\sigma_t^2)}$ is the modulation coefficient.\(^4\)

From Eq. 5.5, one can see that the instantaneous frequency is red-shifted (to lower frequencies) in the leading part of the pulse while being blue-shifted

\(^3\)This concept will be further explained in Ch. 8.3.3.

\(^4\)From here on, the linear refractive index $n_0$ will be denoted as $n$, unless specifically mentioned otherwise.
Figure 5.2: Self phase modulation in (a) temporal domain creating a chirped pulse, (b) spectral domain which broadens the spectrum of the un-chirped pulse.
5.2. Passive mode locking

(to higher frequencies) in the trailing part, creating new frequency components in an overall chirped pulse as seen in Fig.5.2 by the process \cite{36} termed self phase modulation (SPM). We note that it is more convenient to plot the pulse in Fig.5.2(a) by approximating Eq.5.5 to a linear modulation, where $\omega(t) = \omega_0 + \epsilon t$, hence $E(t) = \exp(-t^2/\sigma_t^2) \cos[(\omega_0 + \epsilon t)t]$. The reason is because a linear chirp through the entire pulse is more visually prominent than a linear chirp only to the middle portion of the pulse. However, the spectral broadening plotted in Fig.5.2(b) is calculated by using Eq.5.3 without approximation.
Chapter 6

Basic cavity elements

In the following, we describe the basic and most fundamental elements of an optical cavity:

- Active gain medium
- End mirrors and intra-cavity lenses

6.1 Active gain medium

6.1.1 Absorption and emission spectra

For the last two decades the TiS crystal serves as a "working horse" for tunable CW and mode locked lasers due to its broad emission spectrum. A detailed description regarding the TiS crystal properties can be found in [37, 38], including: molecular/electronic structure, absorption/emission spectra, crystal growth techniques, doping level, figure of merit, etc. The crystal has an excellent mechanical properties and great photo- and thermo-resistance, which makes it very convenient to work with. The sapphire \([Al_2O_3]\) hexagonal
6.1. Active gain medium

crystal contains an Aluminum ions in the center and Oxygen atoms in the vertices. During the crystallization process, it is doped with Titanium \([Ti^{3+}]\) ions replacing some of the Aluminium ions. The result is a four-level laser medium with a broad and widely separated absorption/emission spectra \[39\], as seen in Fig.6.1. The absorption band of the TiS crystal is centered at 490nm, so it can be pumped with blue light Argon-Ion gas lasers, or by the less bulky, frequency doubled Nd:YAG solid-state lasers, using 532nm pump with slightly reduced absorption. In general, to maximize absorption, it is important to set the pump polarization to be parallel to the crystal c-axis.

6.1.2 Figure of merit

During the doping process of the crystal, one cannot avoid a small fraction of parasitic \(Ti^{4+}\) ions which introduce a weak residual absorption of the near infra-red (NIR) wavelengths \[40\], centered between 800nm and 850nm. This NIR absorption is due to \(Ti^{3+} - Ti^{4+}\) ion pairs \[41\], hence it is increased with increasing crystal doping at a faster rate than the increase in the visible-light absorption, until it reaches a maximum at the point where 50% of the ions are \(Ti^{3+}\) and the others are \(Ti^{4+}\). Consequently, the Figure of Merit (FOM) of the crystal is defined as: \(FOM = \frac{\alpha_{\lambda=514}}{\alpha_{\lambda=820}}\), where \(\alpha\) is the absorption coefficient, defined as the decrease of the intensity through the crystal: \(I(x) = I_0 \exp(-\alpha x)\). For a given doping value, one can optimize the FOM based on the crystal growing technique, but as a general rule, the FOM decreases with increasing crystal doping. For example: a 0.15\%wt doped crystal with \(\alpha_{\lambda=514} = 2.8cm^{-1}\) can have \(FOM > 250\), whereas a 0.25\%wt doped crystal with \(\alpha_{\lambda=514} = 4.6cm^{-1}\) will have only \(FOM > 150\).
Figure 6.1: (a) Measured emission spectrum of a TiS crystal. The absorption spectrum is illustrated in the inset graph, (b) schematic diagram of the energy levels in a TiS crystal, operating as a four-level laser system.
6.2 Mirrors and lenses

The basic configuration of a TiS cavity is illustrated in Fig. 6.2. The feedback end mirrors are the high-reflector (HR) and the output-coupler (OC), while two lenses with foci $f_1$ and $f_2$ are used to focus the laser mode into the crystal and to stabilize the cavity. If we define the distance between the lenses as: $f_1 + f_2 + \delta$ ($\delta$ measures the distance off from perfect telescope), one can represent a single round trip in the cavity with respect to a given reference plane as a 2x2 matrix, and use the ABCD matrix technique to analyze the cavity and find the range of values for $\delta$ in which the cavity is stable.

---

1. The ABCD matrix technique for cavity analysis is detailed in Ch.
Chapter 7

ABCD matrix analysis

In the following, we present the ABCD matrix technique, which is widely used to describe the propagation of optical rays through optical elements.

7.1 Vector/matix representation of an optical beam/system

The ABCD matrices are used to represent an optical system as a 2x2 matrix, so one can calculate how the optical system affects the path and properties of a beam of light passing through it. The beam in the system input is represented by a vector $V_{in}$ containing the distance $x_{in}$ above the optical axis and the beam angle $\theta_{in}$ with respect to the optical axis. The beam at the output of the system can be calculated by: $V_{out} = MV_{in}$, where $M$ is the optical system ABCD matrix:

\footnote{The ABCD matrices are formulated within the paraxial approximation, which assumes small values off $x$ and $\theta$. Consequently, expressions containing $\sin \theta$ and $\tan \theta$ are approximated such that: $\sin \theta \approx \tan \theta \approx \theta$.}
7.2. Basic ABCD matrices

\[
\begin{bmatrix}
  x_{\text{out}} \\
  \theta_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  C & D
\end{bmatrix}
\begin{bmatrix}
  x_{\text{in}} \\
  \theta_{\text{in}}
\end{bmatrix}
\] (7.1)

A system containing many optical elements can be described by a single ABCD matrix, which is the multiplication of all the matrices of the elements in the system:

\[
M_{\text{tot}} = \prod_{i=1}^{N} M_i = M_N M_{N-1} \ldots M_2 M_1 \quad (7.2)
\]

The determinant of an ABCD matrix $M$ representing an optical system is:

\[
det(M) = AB - CD = \frac{n_1}{n_2} \quad (7.3)
\]

where $n_1$ and $n_2$ are the refractive indices at the system input and output, respectively. For example, the determinant of an optical system starting inside water and ends inside vacuum will be: $det(M) = 1.33$.

7.2 Basic ABCD matrices

In the following, we will construct the fundamental ABCD matrices required to analyze optical resonators. The ABCD matrices are formulated within the paraxial approximation, which assumes small values of $x$ and $\theta$. We will start with the ABCD matrix that represents a beam propagating through free space with length $L$, as illustrated in Fig. 7.1. The input beam is represented
Figure 7.1: Illustration of a beam propagating a distance $L$ through free space.

by an arbitrary vector: $V_{in} = [x_1, \theta]$. After propagating a distance $L$, the output vector should have the same angle $\theta$ but with a different distance from the optical axis: $V_{out} = [x_2, \theta]$. Using the paraxial approximation, we can relate $x_1$ to $x_2$ by: $x_2 - x_1 = L \tan \theta \approx L \theta$. Using Eq.7.1 we obtain two equations:

$$x_2 = Ax_1 + B \theta$$

$$\theta = C x_1 + D \theta$$

(7.4)

It is clear that the ABCD elements are: $A = 1, B = L, C = 0, D = 1$. Therefore, the free space propagation matrix is:

$$M_{FS} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

(7.5)

Next, we will construct the ABCD matrix that represents a thin lens with focal length $f$. We start with an input beam parallel to the optical
3.2. Basic ABCD matrices

Figure 7.2: Illustration of a beam propagating through a thin lens with focal length $f$.

Axis, hence: $\mathbf{V}_m = [x, 0]$, as illustrated in Fig. 7.2(a). Immediately after the lens, the output beam has the same value of $x$ but with an angle $\theta$ such that the beam will cross the optical axis after a distance $f$ from the lens. Using the paraxial approximation ($\tan \theta \approx \theta$), the angle can be approximated by: $\theta \approx -x/f$. Using Eq. 7.1 we obtain two equations:

$$x = Ax$$
$$-\frac{x}{f} = Cx$$

(7.6)

It is clear that $A = 1$ and $C = -1/f$. In order to find the values of $B$ and
Chapter 7. ABCD matrix analysis

Chapter 7. ABCD matrix analysis

D, we consider an output beam that originated from a point on the optical axis located at a distance $f$ before the lens, as illustrated in Fig. 7.2(b). At the input of the lens, the beam vector is: $\mathbf{V}_{\text{in}} = [x, x/f]$, when the angle has been approximated by: $\tan \theta \approx \theta \approx x/f$. Immediately after the lens, the beam is parallel to the optical axis, hence: $\mathbf{V}_{\text{out}} = [x, 0]$. Using Eq. 7.1 we obtain two equations:

\[
x = x + B \frac{x}{f}
\]

\[
0 = -\frac{1}{f} x + D \frac{x}{f}
\]

(7.7)

It is clear that $B = 0$ and $D = 1$. Therefore, the ABCD matrix representing a thin lens is given by:

\[
\begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix}
\]

(7.8)

7.2.1 Radius of curvature

Using the vector representation of a beam, one can define a radius of curvature for a given beam vector $\mathbf{V} = [x, \theta]$. As illustrated in Fig. 7.3, the radius of curvature of a beam $R$ is related to the beam parameters by: $\sin \theta = x/R$. Using the paraxial approximation, we find:

\[
R = \frac{x}{\theta}
\]

(7.9)

Equation 7.9 results in a sign convention in which $R$ is positive if the curvature is concave towards the light. In other words, $R$ is positive if the beam is propagating away from the optical axis and negative if the beam
Figure 7.3: Illustration of an optical system that transforms the radius of curvature of a beam.

propagates towards the optical axis. If $R_1$ is the radius of curvature at the input of the optical system, one can calculate the radius of curvature at the output of the beam $R_2$:

$$R_2 = \frac{x_2}{\theta_2} = \frac{Ax_1 + B\theta_1}{Cx_2 + D\theta_2} = \frac{AR_1 + B}{CR_1 + D} \quad (7.10)$$

If the optical system is a thin lens represented by Eq.7.8 then Eq.7.10 becomes:

$$\frac{1}{f} = \frac{1}{R_1} - \frac{1}{R_2} \quad (7.11)$$

7.2.2 Zero ABCD elements

Figure 7.4 illustrates different scenarios in which a beam is affected by an optical system represented by an ABCD matrix where one of the elements of the matrix is zero.
• **A = 0.** In this case, the hight $x_2$ of the output beam depends only on the angle of the input beam. Therefore, all beams entering the system with equal angle $\theta_1$ will converge to a single point located at: $x_2 = B\theta_1$.

• **B = 0.** In this case, the hight $x_2$ of the output beam depends only on the hight of the input beam. Therefore, all beams entering the system after leaving a point located at $x_1$, will converge to a single point located at: $x_2 = Ax_1$. It is clear that $A$ plays the role of magnification.

• **C = 0.** In this case, the angle $\theta_2$ of the output beam depends only on the angle of the input beam. Therefore, all beams entering the system with equal angle $\theta_1$, will exit the system with equal angles as well: $\theta_2 = D\theta_1$. It is clear that $D$ plays the role of angular magnification.

• **D = 0.** In this case, the angle $\theta_2$ of the output beam depends only on the hight of the input beam. Therefore, all beams entering the system after leaving a point located at $x_1$, will exit the system parallel to each other, with equal angles: $\theta_2 = Cx_1$.

### 7.3 ABCD matrices for tilted surface

Figure 7.5 illustrates a beam refracting into a tilted spherical interface. The corresponding ABCD matrices for the sagittal and tangential planes are given by:

\[
M_s = \begin{bmatrix}
1 & 0 \\
\frac{n_2 \cos \theta_2 - n_1 \cos \theta_1}{n_2 R} & \frac{n_2}{n_2}
\end{bmatrix} \quad M_t = \begin{bmatrix}
\frac{\cos \theta_2}{\cos \theta_1} & 0 \\
\frac{n_2 \cos \theta_2 - n_1 \cos \theta_1}{n_2 R \cos \theta_1 \cos \theta_2} & \frac{n_1 \cos \theta_1}{n_2 \cos \theta_2}
\end{bmatrix}
\] (7.12)
7.3. ABCD matrices for tilted surface

where $R$ is positive if the surface is concave towards the light. From the matrices in Eq.7.12, one can construct all the other matrices that we will use in order to analyze optical resonators.

The matrices for a flat interface, can be derived from Eq.7.12 by taking the limit of $R \to \infty$:

$$M_s = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix} \quad \quad M_t = \begin{bmatrix} \cos \theta_2 & 0 \\ \frac{n_1 \cos \theta_1}{n_2 \cos \theta_2} & 0 \end{bmatrix}$$

It is clear that for a flat mirror we have: $n_1 = n_2$ and $\theta_1 = \theta_2$, hence both matrices in Eq.7.13 become the unity matrix $I$.

Next, we will construct the ABCD matrices that represent a tilted window, illustrated in Fig.7.6, for the sagittal and tangential planes. We will use the matrices in Eq.7.13 and the matrix for propagation in free space.
Chapter 7. ABCD matrix analysis

Figure 7.5: Illustration of a beam refraction between two media separated by a tilted spherical interface.

Figure 7.6: Beam propagation through a tilted window.
7.3. ABCD matrices for tilted surface

The overall matrix that represents a tilted window, can be constructed by multiplying three matrices: refracting into the material, propagation and refracting out of the material. For the sagittal plane, we obtain:

\[
M_s = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{bmatrix}
\]  
\[
= \begin{bmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{bmatrix}
\]  
(7.14)

and for the tangential plane, we obtain:

\[
M_t = \begin{bmatrix} \cos \theta_3 \cos \theta_4 & 0 & \cos \theta_3 \cos \theta_2 \\ 0 & \frac{n \cos \theta_3}{\cos \theta_4} & \cos \theta_1 \cos \theta_4 \\ \cos \theta_1 \cos \theta_3 & \frac{n \cos \theta_2 \cos \theta_3}{\cos \theta_4} & \cos \theta_2 \cos \theta_4 \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos \theta_3 \cos \theta_4 & 0 & \cos \theta_3 \cos \theta_2 \\ 0 & \frac{n \cos \theta_3}{\cos \theta_4} & \cos \theta_1 \cos \theta_4 \\ \cos \theta_1 \cos \theta_3 & \frac{n \cos \theta_2 \cos \theta_3}{\cos \theta_4} & \cos \theta_2 \cos \theta_4 \end{bmatrix}
\]  
\[
= \begin{bmatrix} \cos \theta_3 \cos \theta_4 & 0 & \cos \theta_3 \cos \theta_2 \\ 0 & \frac{n \cos \theta_3}{\cos \theta_4} & \cos \theta_1 \cos \theta_4 \\ \cos \theta_1 \cos \theta_3 & \frac{n \cos \theta_2 \cos \theta_3}{\cos \theta_4} & \cos \theta_2 \cos \theta_4 \end{bmatrix}
\]  
(7.15)

If the input and output interfaces are parallel to each other, then \( \theta_1 = \theta_4 \) and \( \theta_2 = \theta_3 \), hence the matrix for the tangential plane becomes:

\[
M_t = \begin{bmatrix} 1 & \frac{L}{n} \left( \frac{\cos \theta_3}{\cos \theta_2} \right)^2 \\ 0 & 1 \end{bmatrix}
\]  
(7.16)

Specifically, if the incidence angle \( \theta_1 \) is the Brewster angle, then: \( \theta_1 + \theta_2 = 90^\circ \). Using Snell’s law \( \sin \theta_1 = n \sin \theta_2 \), we obtain: \( \cos \theta_2 = n \cos \theta_1 \).

Substituting into Eq(7.16) yields:

\[
M_t = \begin{bmatrix} 1 & \frac{L}{n} \\ 0 & 1 \end{bmatrix}
\]  
(7.17)

The two matrices in Eq(7.14) and Eq(7.17) can also be used even if the input and output interfaces are not parallel to each other, for example: a
prism in which the apex angle \( \alpha \) satisfies: \( \alpha = \theta_2 + \theta_3 \). If \( \alpha \) is chosen such that: \( \alpha = 2\theta_2 \), hence: \( \theta_2 = \theta_3 \) and \( \theta_1 = \theta_4 \). If the incidence angle \( \theta_1 \) is the Brewster angle, we also obtain: \( \cos \theta_2 = n \cos \theta_1 \), hence one can use the above specified matrices as well.

Using the matrices in Eq.7.12 and Eq.7.13 we can construct the ABCD matrices for the sagittal and tangential planes, that represent a tilted thin lens with refractive index \( n \) and a focal length \( f_0 \) for normal incidence. This can be achieved by assuming a plano-convex (or plano-concave) lens, which can be represented by the multiplication of two matrices: one for a tilted spherical interface and another for a flat interface, while using the relation: \( \theta_2 = \theta_3 \) and \( \theta_1 = \theta_4 \). For the sagittal plane we obtain:

\[
M_s = \begin{bmatrix}
1 & 0 \\
0 & n \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{n \cos \theta_2 - \cos \theta_1}{nR} & \frac{1}{n} \\
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\frac{n \cos \theta_2 - \cos \theta_1}{R} & 1 \\
\end{bmatrix}
\]

(7.18)

The \( C \) element in the matrix is related to the focal length by: \( C = -f_s^{-1} \), as given by Eq.7.8. From the lens maker equation\(^2\) we know that \( R = -f_0(n - 1) \). Substituting into the \( C \) element in the matrix, we can obtain the focal length of a tilted lens for the sagittal plane:

\[
f_s = \frac{f_0(n - 1)}{n \cos \theta_2 - \cos \theta_1}
\]

(7.19)

\(^2\)The lens maker equation: \( f_0^{-1} = (n - 1)(R_2^{-1} - R_1^{-1}) \), where the sign convention is that the \( R_i \) is positive if the interface is concave towards the light.
In the same way one can obtain the focal length of a tilted lens for the tangential plane:

\[
M_t = \begin{bmatrix}
\frac{\cos \theta_1}{\cos \theta_2} & 0 \\
0 & \frac{n \cos \theta_2}{\cos \theta_1}
\end{bmatrix}
\begin{bmatrix}
\frac{\cos \theta_2}{\cos \theta_1} & 0 \\
\frac{n \cos \theta_2 - \cos \theta_1}{n \cos \theta_1 \cos \theta_2} & \frac{\cos \theta_1}{n \cos \theta_2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
\frac{n \cos \theta_2 - \cos \theta_1}{R \cos \theta_1} & 1
\end{bmatrix}
\]

(7.20)

Here also, the \( C \) element in the matrix is related to the focal length by:

\[ C = -\frac{1}{f_t} \]. Using the lens maker equation and substituting into the \( C \) element in the matrix, we can obtain the focal length of a tilted lens for the tangential plane:

\[ f_t = \frac{f_0(n - 1) \cos^2 \theta_1}{n \cos \theta_2 - \cos \theta_1} \quad (7.21) \]

For a curved mirror with focal length \( f_0 \), we have \( \theta_1 = \theta_2 = \theta \), hence Eq.7.21 and Eq.7.19 becomes:

\[ f_s = \frac{f_0}{\cos \theta} \quad f_t = f_0 \cos \theta \quad (7.22) \]

7.4 Stability condition of an optical system

The effects of an optical system represented by a matrix \( M \) can be understood by calculating the eigenvalues and eigenvectors of the matrix.

7.4.1 Eigenvalues

The eigenvalues \( \lambda_i \) are given by\(^3\)

\[ \det(M - \lambda I) = 0 \quad (7.23) \]

\(^3\)The eigenvalues \( \lambda_i \) are not related to the wavelength in any way.
where \( I \) is the unity matrix. If we assume that the refractive indices are equal at the input and output of the matrix \( \mathbf{M} \), we obtain: \( \text{det}(\mathbf{M}) = AB - CD = 1 \).

Therefore, Eq.\ref{eq:7.23} can be written as:

\[
\lambda^2 - (A + D)\lambda + 1 = 0
\] (7.24)

The solution for the eigenvalues is:

\[
\lambda_{\pm} = \frac{A + D}{2} \pm \sqrt{\left(\frac{A + D}{2}\right)^2 - 1}
\] (7.25)

where it is clear that the eigenvalues satisfy: \( \lambda_+\lambda_- = 1 \).

### 7.4.2 Eigenvectors

The eigenvectors of the matrix \( \mathbf{M} \) satisfy:

\[
\mathbf{M}\mathbf{s}_1 = \lambda_+\mathbf{s}_1 \quad \mathbf{M}\mathbf{s}_2 = \lambda_-\mathbf{s}_2
\] (7.26)

If we define \( \mathbf{S} \) as the matrix of the eigenvectors: \( \mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2] \), Eq.\ref{eq:7.26} can be written in a matrix form:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{bmatrix}
= \begin{bmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{bmatrix}
\begin{bmatrix}
\lambda_+ & 0 \\
0 & \lambda_-
\end{bmatrix}
\]

(7.27)

Solving Eq.\ref{eq:7.27}, we find that the eigenvectors are given by:

\[
\mathbf{s}_1 = \begin{bmatrix}
s_{11} \\
s_{21}
\end{bmatrix}
= \begin{bmatrix}
\lambda_+ \\
C
\end{bmatrix}
\]
\[
\mathbf{s}_2 = \begin{bmatrix}
s_{12} \\
s_{22}
\end{bmatrix}
= \begin{bmatrix}
\lambda_- \\
C
\end{bmatrix}
\] (7.28)
Now that we have found the eigenvalues and eigenvectors of the matrix $\mathbf{M}$, we know that any vector $\mathbf{V} = [v_1, v_2]$ that represents a beam, can be represented as a linear combination of the eigenvectors of matrix $\mathbf{M}$, such that:

$$\mathbf{V} = c_1 \mathbf{S}_1 + c_2 \mathbf{S}_2$$  \hfill (7.29)

where $c_1$ and $c_2$ are coefficients that can be calculated by:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$  \hfill (7.30)

Hence, the output vector $\mathbf{V}_{out}$ after the beam passes through the optical system is:

$$\mathbf{V}_{out} = \mathbf{M} \mathbf{V}_{in} = c_1 \mathbf{M} \mathbf{S}_1 + c_2 \mathbf{M} \mathbf{S}_2 = c_1 \lambda_+ \mathbf{S}_1 + c_2 \lambda_- \mathbf{S}_2$$  \hfill (7.31)

Now we assume that we are dealing with a periodic optical system, in which the beam passes through the system over and over again. This is the case in optical resonators, in which a single round trip in the resonator can be presented by an ABCD matrix and the beam circulates through the resonator. Therefore, if the beam starts with a vector $\mathbf{V}_{in}$, after $N$ round trips through the resonator, the output vector will be:

$$\mathbf{V}_{out} = \mathbf{M}^N \mathbf{V}_{in} = c_1 \lambda_+^N \mathbf{S}_1 + c_2 \lambda_-^N \mathbf{S}_2 = c_1 \lambda_+^N \mathbf{S}_1 + c_2 \lambda_-^N \mathbf{S}_2$$  \hfill (7.32)

where the realtion $\lambda_+ \lambda_- = 1$ have been used.
Chapter 7. ABCD matrix analysis

7.4.3 Unstable systems

From Eq.7.25, it is clear that if $|A + D| > 2$, then $|\lambda_+| > 1$. From Eq.7.31, it is clear that $\lambda_+$ plays the role of magnification. Therefore, the beam will diverge exponentially by a factor of $\lambda_+^N$ after the beam circulates $N$ round trips in the resonator. There is also a converging term proportional to $\lambda_-^N$, but it will quickly decay after a few round trips. Hence, the resonator is considered unstable if $|A + D| > 2$.

7.4.4 Stable systems

Let us consider the case where $|A + D| < 2$. Equation 7.25 can be written as:

$$\lambda_\pm = \frac{A + D}{2} \pm i\sqrt{1 - \left(\frac{A + D}{2}\right)^2} = e^{\pm i\theta}$$

Therefore, the output vector will be:

$$V_{out} = M^N V_{in} = c_1\lambda_+^N S_1 + c_2\lambda_-^N S_2 = c_1 e^{iN\theta}S_1 + c_2 e^{-iN\theta}S_2$$

Now the beam does not diverge as it circulates through the resonator, hence the condition for resonator stability is:

$$|A + D| < 2$$
Chapter 8

Cavity analysis

In the following, we will describe the stability zones of an optical cavity. In addition, we will analyze the cavity in CW and ML operations.

8.1 Stability zones

The cavity stability condition is given in Eq. (7.35) where $A$ and $D$ are the elements of an ABCD matrix that represents a single round trip in the cavity. In the following we will analyze the stability regimes of operation in two basic laser configurations.

8.1.1 Ring resonator

Figure 8.1 illustrates an optical resonator in ring configuration, in which the beam passes through every optical element in the cavity only once in each round trip. The ABCD matrix that represents a single round trip in the
cavity, can be calculated as:

\[
M_{rt} = \begin{bmatrix}
1 & L_0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\frac{1}{f_2} & 1
\end{bmatrix}
\begin{bmatrix}
1 & f_1 + f_2 + \delta \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\frac{1}{f_1} & 1
\end{bmatrix}
\begin{bmatrix}
1 & L_0 \\
0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{L_0 \delta}{f_1 f_2} - \frac{\delta + f_2}{f_1} & \cdots \\
\cdots & \frac{L_0 \delta}{f_1 f_2} - \frac{\delta + f_1}{f_2}
\end{bmatrix}
\]

(8.1)

where only the \(A\) and \(D\) elements are displayed, since the stability condition of the cavity in Eq. 7.35 does not depend on the elements \(B\) and \(C\).

Using \(|A + D| < 2\), we find that the ring resonator is stable in a range of \(\delta\) values such that \(\delta_0 < \delta < \delta_1\), given by:

\[
\delta_0 = \frac{(f_1 - f_2)^2}{2L_0 - (f_1 + f_2)} \quad \delta_1 = \frac{(f_1 + f_2)^2}{2L_0 - (f_1 + f_2)}
\]

(8.2)
8.1. Stability zones

8.1.2 Linear resonator

A resonator in linear configuration is illustrated in Fig.6.2. In order to construct the round trip ABCD matrix, we need to multiply two matrices, one representing the propagation from \( EM_1 \) to \( EM_2 \), and the other from \( EM_2 \) to \( EM_1 \). The matrices are given by:

\[
M_{12} = \begin{bmatrix}
1 & L_2 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{-1}{f_2} & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & f_1 + f_2 + \delta \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{-1}{f_1} & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & L_1 \\
0 & 1 \\
\end{bmatrix}
\]

\[
M_{21} = \begin{bmatrix}
1 & L_1 \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{-1}{f_1} & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & f_1 + f_2 + \delta \\
0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
\frac{-1}{f_2} & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & L_2 \\
0 & 1 \\
\end{bmatrix}
\]

(8.3)

The round trip matrix is given by: \( M_{rt} = M_{21}M_{12} \). The \( A \) and \( D \) elements in \( M_{rt} \) are equal and given by:

\[
A = D = 2L_1L_2 - L_1f_2 - L_2f_1 + f_1f_2\delta^2 + 2f_1^2f_2 + f_1f_2^2 - L_2f_1^2 + L_1f_2^2 \]

\[
+ \frac{(f_1f_2)^2}{(f_1f_2)^2} \delta + 1
\]

(8.4)

Using \( |A + D| < 2 \), one finds that the laser is stable for two bands of \( \delta \) values bounded between four stability limits: \( \delta_0 < \delta < \delta_1 \) and \( \delta_2 < \delta < \delta_3 \), given by:

\[
\delta_0 = 0, \quad \delta_1 = \frac{f_2^2}{(L_2 - f_2)}, \quad \delta_2 = \frac{f_1^2}{(L_1 - f_1)}, \quad \delta_3 = \delta_1 + \delta_2
\]

(8.5)

For simplification, Eq.8.5 and Eq.8.2 were calculated without the TiS crystal between \( f_1 \) and \( f_2 \). The addition of TiS crystal (or any material with thickness \( L \) and refractive index \( n \)) will shift all the stability limits \( \delta_i \) by a constant factor of \( C = L(1 - 1/n) \). However, Eq.8.5 can still be used without
any change by redefining the distance between $f_1$ and $f_2$ in Fig.6.2 to be: $f_1 + f_2 + C + \delta$.

Near the stability limits given in Eq.8.5, the stable cavity mode can be intuitively visualized using geometrical optics, as illustrated in Fig.8.2. The four stability limits can be named according to the mode size behaviour on the cavity end mirrors, as follows: 1. plane-plane limit [Fig.8.2(a)], the lenses are separated by $f_1 + f_2$, forming a perfect telescope which produces a collimated beam in both arms; 2. plane-point limit [Fig.8.2(b)], the lenses are separated by $f_1 + f_2 + \delta_1$ such that the focal point between the lenses is imaged on $EM_2$; 3. point-plane limit [Fig.8.2(c)], the lenses are separated by $f_1 + f_2 + \delta_2$ and the focal point between the lenses is imaged on $EM_1$; 4. point-point limit [Fig.8.2(d)], the lenses are separated by $f_1 + f_2 + \delta_3$ and the focus point between the lenses is imaged on both the end mirrors. Kerr lens mode locking usually occurs near one of the stability limits, as explained in Ch.8.3.2.

8.2 Cavity analysis in continuous-wave operation

8.2.1 Gaussian modes

In order to analyze the cavity in CW operation, one needs to find a solution to the wave equation, which can be simplified under paraxial approximations into the "paraxial Helmholtz equation", given by:
8.2. Cavity analysis in continuous-wave operation

\[ \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - 2ik \frac{\partial \Psi}{\partial z} = 0 \]  

(8.6)

where \( \Psi(x, y, z) \) is related to the electric field by: \( E = \Psi \exp[i(\omega t - kz)] \).

As a solution to Eq. (8.6), we represent \( \Psi \) as a general complex number with amplitude and phase, in terms of another two functions, \( \psi(z) \) and \( q(z) \):

\[ \Psi(x, y, z) = \psi(z) \exp \left[ -i \frac{k(x^2 + y^2)}{2q(z)} \right] \]  

(8.7)

Substituting Eq. (8.7) into Eq. (8.6) yields:

\[ \frac{k^2(x^2 + y^2)}{q^2} \left( 1 - \frac{\partial q}{\partial z} \right) + 2ik \left( \frac{1}{q} + \frac{1}{\psi} \frac{\partial \psi}{\partial z} \right) = 0 \]  

(8.8)
Chapter 8. Cavity analysis

From Eq. 8.8 we obtain two equations for \( q(z) \) and \( \psi(z) \). The first equation for \( q(z) \) is given by:

\[
\frac{\partial q}{\partial z} = 1 \tag{8.9}
\]

which yields the solution: \( q(z) = z + q_0 \), where \( q_0 = q(z = 0) \) is an integration constant. The second equation for \( \psi(z) \) is given by:

\[
\frac{\partial \psi}{\partial z} = -\frac{\psi q}{z + q_0} \tag{8.10}
\]

which yields the solution:

\[
\psi(z) = \frac{\psi_0 q_0}{q(z)} \tag{8.11}
\]

Therefore, Eq. 8.7 becomes:

\[
\Psi(x, y, z) = \frac{\psi_0 q_0}{z + q_0} \exp \left[ -i \frac{k}{2} \frac{x^2 + y^2}{z + q_0} \right] \tag{8.12}
\]

It is convenient to represent \( q(z) \) in terms of two real valued functions: \( R(z) \) and \( \omega(z) \), given by:

\[
\frac{1}{q(z)} = \frac{1}{R(z)} - i \frac{\lambda}{\pi \omega^2(z)} \tag{8.13}
\]

where \( \lambda \) is the wavelength inside the material, given by: \( \lambda = \lambda_0 / n \), where \( \lambda_0 \) is the wavelength in vacuum and \( n \) is the refractive index. In the following, we will find that \( R(z) \) and \( \omega(z) \) represent the radius of curvature of the beam wavefront and the beam radius as a function of \( z \), respectively. From Eq. 8.13
we find that $\Psi$ has a radial Gaussian distribution both in amplitude and phase:

$$\Psi \sim \exp \left( -\frac{r^2}{\omega^2} - i \frac{k r^2}{2 R} \right)$$  \hspace{1cm} (8.14)

The radial phase becomes zero as $R \to \infty$. We set the point of zero phase to be at $z = 0$, hence the integration constant $q_0$ can be calculated from Eq.8.13:

$$q_0 = i \frac{\pi \omega_0^2}{\lambda} = iz_R$$  \hspace{1cm} (8.15)

where $\omega_0$ is the beam radius at $z = 0$ and $z_R = \pi \omega_0^2 / \lambda$ is called the Raleigh range of the beam. Hence, the parameter $q(z)$ is called the complex beam parameter and is given by:

$$q = z + iz_R$$  \hspace{1cm} (8.16)

Calculating $q^{-1}$ from Eq.8.16 and equating with Eq.8.13 yields:

$$\frac{1}{q} = \frac{1}{z + iz_R} = \frac{z}{z^2 + z_R^2} - i \frac{z_R}{z^2 + z_R^2} = \frac{1}{R} - i \frac{\lambda}{\pi \omega^2}$$  \hspace{1cm} (8.17)

By equating the real and imaginary parts in Eq.8.17, one can calculate $R(z)$ and $\omega(z)$:

$$R(z) = z \left[ 1 + \left( \frac{z_R}{z} \right)^2 \right], \quad \omega^2(z) = \omega_0^2 \left[ 1 + \left( \frac{z}{z_R} \right)^2 \right]$$  \hspace{1cm} (8.18)

The divergence angle of the beam $\theta$ can be calculated using the relation: $\tan \theta = \omega(z)/z$ at limit where $z \to \infty$, given by:
\[ \tan \theta = \frac{\omega(z)}{z} = \frac{\omega_0}{z} \sqrt{1 + \left( \frac{z}{z_R} \right)^2} = \omega_0 \sqrt{\frac{1}{z^2} + \left( \frac{1}{z_R} \right)^2} = \frac{\lambda}{\pi \omega_0} \quad (8.19) \]

The complex function \( \psi \) can also be expressed as a real amplitude and a phase:

\[ \psi(z) = \psi_0 \frac{izR}{z + izR} = \frac{\omega_0}{\omega(z)} \exp(i\phi) \quad (8.20) \]

where the phase term is called Gouy phase, given by: \( \tan \phi = z/z_R \).

Finally, the electric field is given by:

\[ E(r, z) = E_0 \frac{\omega_0}{\omega(z)} \exp \left( -\frac{r^2}{\omega^2(z)} \right) \exp \left( -i \frac{k r^2}{2R(z)} + i\phi \right) \exp[i(\omega t - kz)] \]

\[ (8.21) \]

where \( \psi_0 \) was denoted as \( E_0 \) and \( r^2 = x^2 + y^2 \). Note the difference between the width of the beam \( \omega(z) \) and the frequency \( \omega \). A cross section of a Gaussian beam representing the beam radius \( \omega \) as a function of the beam propagation axis \( z \) is illustrated in Fig.8.3.

**8.2.2 Stable Gaussian solutions**

Using ABCD matrices one can calculate the fundamental Gaussian \( TEM_{00} \) mode of the cavity for CW operation inside the stability zones of the resonator. The Gaussian mode is represented by the complex beam parameter \( q = z + iz_R \). The relation between the complex beam parameter before an optical system \( q_1 \) and after the system \( q_2 \) obeys the same relation given is Eq.7.10.
8.2. Cavity analysis in continuous-wave operation

Figure 8.3: Gaussian beam width $\omega(z)$ as a function of the axial distance $z$.

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \tag{8.22}$$

To calculate $q$ at any reference plane in the cavity, one needs to represent the complete single round trip in the cavity as an ABCD matrix with respect to the reference plane. The solution of the complex beam parameter of the cavity mode at the reference plane is given by the solution of the following quadratic equation:

$$Cq_{cw}^2 + (D - A)q_{cw} - B = 0 \tag{8.23}$$

Using the relation $AD - CB = 1$, Eq.8.23 can be rewritten for the inverse complex beam parameter:

$$\left(\frac{1}{q_{cw}}\right)^2 + \frac{A - D}{B} \frac{1}{q_{cw}} + \frac{1 - AD}{B^2} = 0 \tag{8.24}$$

The solution to Eq.8.24 is given by:

$$\left(\frac{1}{q_{cw}}\right)_\pm = \frac{D - A}{2B} \pm \frac{i}{B} \sqrt{1 - \left(\frac{A + D}{2}\right)^2} \tag{8.25}$$
Figure 8.4: CW mode size on the end mirrors $EM1$ and $EM2$ for the cavity illustrated in Fig.6.2 with $f_1 = f_2 = 7.5cm$ and arm-lengths of $L_1 = 30cm$ (short arm) and $L_2 = 40cm$ (long arm). At $\delta_1$ the beam is collimated in the short arm and non-collimated in the long arm. This behaviour is reversed at $\delta_2$.

Using Eq.8.13 and Eq.8.25, we obtain expressions for the beam spot size $\omega$ and the radius of curvature $R$, given by:

$$R = \frac{2B}{D - A}$$

$$\omega^2 = \frac{|B|\lambda}{\pi} \sqrt{\frac{1}{1 - (A + D)^2/4}}$$  \hspace{1cm} (8.26)

A natural location to calculate the complex beam parameter is at the end mirrors, where the mode must arrive with a plane phase front, hence $q_{CW} = iz_R$ is pure imaginary and the mode waist radius $\omega_0^{(CW)}$ can be plotted as a function of $\delta$, as in Fig.8.4.
8.2.3 Linear astigmatism

In order to minimize losses in the cavity, the active gain medium is positioned at Brewster angle, so that a linearly $p$-polarized laser beam will suffer minimal reflection as it enters/exits the crystal. If we define the tangential plane by the incident and refracted beams and the sagittal plane perpendicular to it, we find that mode size in the tangential plane expands as the beam refracts into the crystal, while the mode size in the sagittal plane remains the same\footnote{In Fig 6.2 the cavity is illustrated in top view of the optical table and the tangential plane is parallel to the optical table.}. This introduces an aberration known as \textit{astigmatism}, which is caused since the beam experiences a different path in each plane. The beam path is given by: $L_s = L/n$ for the sagittal plane and $L_t = L/n^3$ for the tangential plane, where $L$ is the thickness of the active medium and $n$ is the refractive index. This aberration splits the stability limits of the two planes differently from the non-astigmatic stability limits in Eq 8.5, resulting in different stability limits for each plane, as seen in Fig 8.5.

8.2.4 Compensation for linear astigmatism

In order to correct for material astigmatism, one can introduce curved mirrors instead of lenses, resulting in a more realistic configuration of a TiS cavity, illustrated in Fig 8.6. The curved mirrors now fold the cavity in-plane of the optical table and the folding angles introduce additional astigmatism from the curved mirror, which can be used to compensate for material astigmatism of the Brewster windows. The curved mirrors astigmatism is a result of the dependence of the mirror focal length on the reflection angle $\theta$, given
Figure 8.5: CW mode size on end mirror $EM1$ for the cavity with the same parameters as in Fig 8.4 with the addition of 3 $mm$ long Brewster-cut TiS crystal. Due to the crystal astigmatism, all the stability limits of the sagittal plane is pushed towards higher values of $\delta$ while the tangential stability limits are pushed towards lower values (with respect to the stability limits of the non-astigmatic cavity).
8.2. Cavity analysis in continuous-wave operation

Figure 8.6: Folded cavity configurations: (a) X-fold and (b) Z-fold configurations of TiS cavity for linear astigmatism compensation. The folding angles $\theta_1$ and $\theta_2$ introduce curved mirror astigmatism which compensates for the astigmatism of the Brewster-cut TiS crystal.

by: $f_s(f, \theta) = f / \cos \theta$ for the sagittal plane and $f_t(f, \theta) = f \cdot \cos \theta$ for the tangential plane [42]. Therefore, the mirrors astigmatism $\Delta f(f, \theta) = f_s(f, \theta) - f_t(f, \theta)$ can be used to compensate for the crystal astigmatism $\Delta L = L_s - L_t$, by solving:

$$\Delta f(f_1, \theta_1) + \Delta f(f_2, \theta_2) = \Delta L \quad (8.27)$$

For $\theta_1 = \theta_2 = \theta$ and $f_1 = f_2 = f$, one can obtain analytical expression solving Eq.8.27 Given by [43]:

$$\theta = \arccos(\sqrt{1 + N^2} - N) \quad (8.28)$$

where $N$ is given by:

$$N = \frac{L}{4fn} \left(1 - \frac{1}{n^2}\right) \quad (8.29)$$
Solving Eq. 8.27 will compensate only for the stability limit $\delta_0$. This is because the above expression $\Delta f(f, \theta)$ assumes a collimated beam in both arms of the cavity, which is only true at $\delta_0$, as seen in Fig. 8.4. For any other stability limit, one must use a generalized expression for the curved mirrors astigmatism that takes into account non-collimated beams with a point source located at the end mirrors, given by:

$$v_s(f, \theta, D) = \frac{D f_s}{D - f_s} \quad v_t(f, \theta, D) = \frac{D f_t}{D - f_t}$$

(8.30)

Using Eq. 8.30, one can compensate for linear astigmatism for each stability limit $\delta_i$ separately, by solving:

$$\Delta v(f_1, \theta_1, D1_i) + \Delta v(f_2, \theta_2, D2_i) = \Delta L$$

(8.31)

The corresponding values of $D1_i$ and $D2_i$ are given in Table 8.1.

<table>
<thead>
<tr>
<th>$\delta_i$</th>
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<th>$D2_i$</th>
</tr>
</thead>
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<td>$L_2$</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>$L_1$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\delta_3$</td>
<td>$L_1$</td>
<td>$L_2$</td>
</tr>
</tbody>
</table>

Table 8.1: Astigmatism compensation values of the long and short cavity arms for each stability limit.
8.3 Cavity analysis in mode-locked operation

8.3.1 A Gaussian beam in Kerr medium

The additional Kerr-lens of the laser mode due to the optical Kerr effect (Eq.5.2) must be taken into account in order to calculate the Gaussian mode for ML operation. The presence of a lens-like effect inside the crystal, similar to thermal lensing [44], dramatically changes the stability behaviour of the cavity. Several methods have been proposed to solve $q_{ML}$ for ML operation, using a single nonlinear ABCD matrix to treat the propagation of Gaussian beams in materials with Kerr nonlinearity [45, 46]. Another direct approach to solve $q_{ML}$ is to divide the TiS crystal to many thin slices and represent the crystal as a stack of many matrices, each matrix $M_{\text{slice}}$ consists includes propagation through a single slice with thickness $dz$ with an additional nonlinear lens:

$$M_{\text{slice}} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & dz \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & dz \\ -\frac{1}{f} & 1 - \frac{dz}{f} \end{bmatrix} \approx \begin{bmatrix} 1 & dz \\ -\frac{1}{f} & 1 \end{bmatrix}$$

(8.32)

where $f$ is the nonlinear lens given by Eq.5.2. These methods are iterative in nature, since the nonlinear focal length $f$ depends on the mode size. Therefore, in order to solve $q_{ML}$ for a given peak power of the pulse $P$, one starts with the solution of $q_{CW}$ (at $P = 0$) and then calculates the nonlinear

\[\text{Considerably smaller than the Rayleigh range of the laser mode.}\]
Chapter 8. Cavity analysis

stack of ABCD matrices based on the mode size in the crystal given by $q_{\text{CW}}$. The stack defines a new resonator, which yields a solution of a new resonator mode. This procedure continues until a steady state solution is achieved.

It was shown, that dividing the crystal to many thin lens-like slices, can be represented as a differential equation for the inverse complex beam parameter $q^{-1}$, given by [47]:

$$
\frac{d}{dz} \left( \frac{1}{q} \right) + \left( \frac{1}{q} \right)^2 + \frac{d}{dz} \left( \frac{1}{f} \right) = 0 \quad (8.33)
$$

Substituting Eq.8.13 and Eq.5.2 into Eq.8.33 yields:

$$
\frac{d}{dz} \left( \frac{1}{q} \right) + \left( \frac{1}{q} \right)^2 + K \cdot \text{Im}^2 \left( \frac{1}{q} \right) = 0 \quad (8.34)
$$

The nonlinear Kerr lensing process now depends on a normalized parameter: $K = P/P_c$, where $P$ is the pulse peak power and $P_c$ is the critical power for catastrophic self-focusing, given by [5]

$$
\frac{1}{P_c} = \frac{4n_2\pi}{\lambda^2} \quad (8.35)
$$

For $K = 0$, Eq.8.34 reduces back into propagation of a Gaussian beam in free-space, or in any material with linear refractive index $n$.

We can define a new complex beam parameter, $\tilde{q}$, given by:

$$
\frac{1}{\tilde{q}} = \text{Re} \left( \frac{1}{q} \right) + i \cdot \text{Im} \left( \frac{1}{q} \right) \sqrt{1 - K} \quad (8.36)
$$

$^3$Since the nonlinear process depends on the normalized parameter $K$, different theoretical and experimental values for $P_c$ can be defined as well [48].
8.3. Cavity analysis in mode-locked operation

One can show that by multiplying the imaginary part of $q^{-1}$ by: $\sqrt{1 - K}$, the propagation of the new complex beam parameter $\tilde{q}^{-1}$ through a Kerr medium can be reduced back into a free-space propagation:

$$\frac{d}{dz} \left( \frac{1}{\tilde{q}} \right) + \left( \frac{1}{\tilde{q}} \right)^2 = 0 \quad (8.37)$$

At the end of the Kerr medium, $\tilde{q}^{-1}$ is re-transformed by multiplying the imaginary part of $\tilde{q}^{-1}$ by: $\sqrt{1 - K^{-1}}$. Since there is no ABCD matrix that represents this transformation, one can no longer represent the entire cavity by a single ABCD matrix. $q_{ML}$ of the stable mode must thus be obtained by a numerical solution of the stability condition $|A + D| < 2$ for the general cavity configuration illustrated in Fig.8.6. Analytical solutions, however, can still be achieved for a ring cavity [49] or for a symmetrical linear cavity [50] (equal cavity arms). The result is a somewhat different solution for $\omega_0^{(ML)}$ as a function of $\delta$ compared to $\omega_0^{(CW)}$, as seen in Fig.8.7.

It is important to note that the above method ([47] and Eq.8.34) assumes a circular mode inside the crystal and does not include the effects of the more realistic elliptical (astigmatic) mode in the Brewster cut crystal. One may try to calculate $q_{ML}$ for each plane separately by assuming a circular mode with the corresponding mode size ($w_s/w_t$ for the sagittal/tangential plane, respectively), but this is oversimplified in most cases. While such a simple separation of the planes may produce a qualitative understanding of the mode, it cannot provide a quantitative solution since the planes in reality are coupled by the Kerr lensing effect (the mode size in one plane affects the peak intensity, which then affects the mode of the other plane, as explained in Ap.A). A detailed description of the coupled propagation equations of a
Gaussian beam through a Kerr medium is provided in Ap. B and [51].

In addition, all calculations are performed within the aberration-free approximation for the Kerr lens, in which the transverse variation of the refractive index is approximated to be parabolic, so that the beam maintains its Gaussian shape during propagation and the ABCD method to analyse the cavity can be applied. A detailed discussion regarding the limits of the aberration free approximation can be found in [52].

The solution for the ML mode size presented in Fig. 8.7 highly depends on the TiS crystal position. Changing the position of the crystal along the beam affects the mode intensity in the crystal and hence the nonlinear response, leading to a different (sometimes dramatically) solution of $\omega_0^{(ML)}$. In Fig. 8.7(a) the crystal was positioned at the CW focus point between the curved mirrors, whereas in Fig. 8.7(b) the crystal is located away from focus, showing completely opposite behaviours. While in Fig. 8.7(a) the Kerr lens pushes the stability limit for ML operation towards higher values of $\delta$, in Fig. 8.7(b) the stability limit is pulled towards lower values. The position of the crystal accordingly determines which method will be employed in order to mode-lock the cavity - hard or soft aperture, as will be discussed in Ch. 8.3.2.

8.3.2 Hard/soft aperture mode locking

The variation of $\omega_0^{(ML)}$ compared to $\omega_0^{(CW)}$ in Fig. 8.7 can be exploited to introduce an intensity dependent loss mechanism into the cavity. Since the Kerr effect is intensity dependent (Eq. 5.1), it will be maximized near the
8.3. Cavity analysis in mode-locked operation

Figure 8.7: ML and CW mode size on the end mirror EM1 (short arm) near the CW stability limit $\delta_1$ for the cavity illustrated in Fig.8.6. (a) The crystal is at the CW focus with pulse parameter $K \approx 0.1$; the ML stability limit is pushed towards higher values of $\delta$ by the Kerr lens. In (b), the crystal is away from focus with pulse parameter $K \approx 0.2$; the ML stability limit is pulled down by the Kerr lens to lower values of $\delta$. The short-dashed lines (black) represent positions where hard and soft aperture mode locking techniques can be employed (these techniques are discussed in detail in Sec.8.3.2). The long-dashed lines (blue) represent the second stability limit in CW operation. Note in addition, that astigmatism was neglected for simplicity of presentation.
stability limits of the cavity where the laser mode is tightly focused into the crystal. At a given $\delta$ close to the stability limit, the crystal position can be translated along the beam propagation axis to modulate the Kerr strength, given by:

$$\gamma = \frac{P_c}{\omega} \left( \frac{d\omega}{dP} \right)_{P=0},$$

where $P$ is the pulse peak power and $\omega$ is the mode radius at the end mirror. From Eq. 8.38 it follows that the Kerr strength is represented by the change of the CW mode size due to a small increase in the intra-cavity peak power. Plotting $\gamma$ as a function of the crystal position, one finds two possible mechanisms in which loss can be employed upon the CW mode: **Hard aperture**, which requires a physical (or virtual) aperture to block the margins of the CW spatial mode, favoring the smaller pulsed mode; **Soft aperture**, in which the finite pump mode in the crystal is used as an effective aperture to favor pulses.

Figure 8.8 plots $\gamma$ on the end mirror of the short arm ($EM1$) as a function of the crystal position near the end of the first stability zone, $\delta \lesssim \delta_1$. When the crystal is located near focus, $\gamma$ is negative, hence the Kerr lensing decreases the mode size for ML operation compared to CW operation, as seen in Fig. 8.7(a). This enables the introduction of a physical ("hard") aperture at the end mirror $EM1$ to employ loss on the CW mode, while the smaller ML mode passes through the aperture without attenuation. In addition, the Kerr lensing effect in the tangential plane is weaker compared to the sagittal plane in Fig. 8.8 due to the expansion of the mode size in the tangential plane.

---

4This will be discussed in detail in Ch. 8.3.3.
Figure 8.8: Kerr strength $\gamma$ at $EM1$ as a function of the crystal position $Z$ near the stability limit $\delta_1$. At $Z = 0$ the crystal center is located at a distance $f_1$ from $M1$. Positive values of $Z$ correspond to translation of the crystal away from $M1$. Regardless of the crystal position, the sagittal plane has the higher value of $|\gamma|$.
as the beam refracts into the crystal, which reduces the intensity\[5\].

As the crystal is translated away from focus towards $M2$, $\gamma$ becomes positive, hence increasing the mode size for ML operation compared to CW, as seen in Fig.8.7(b). Therefore, a physical aperture cannot be used at $EM1$. However, a larger mode size for ML at $EM1$ corresponds to a smaller mode size at the crystal compared to the CW mode. This enables the pump beam to be used as a "soft" aperture, by setting the pump mode to overlap the smaller ML mode at focus. Consequently, the pump mode acts as an aperture, in which CW modes larger than the aperture enjoy poor pump-mode overlap compared to the ML mode. As can be seen from Fig.8.8 the soft aperture technique for mode locking near $\delta_1$ is less efficient due to the lower value of $|\gamma|$ and also because the translation of the crystal away from focus increases the laser threshold. The efficiency of soft aperture compared to hard aperture mode locking is reversed when mode locking near the beginning of the second stability zone. Figure 8.9 plots $\gamma$ as a function of the crystal position on the end mirror of the long arm ($EM2$) at $\delta \gtrsim \delta_2$.

### 8.3.3 Passive hard aperture mode locking

Maybe the most useful technique for hard aperture mode locking near $\delta_1$ without using a physical aperture, is to choose the mode locking point at $\delta > \delta_1$. As can be seen in Fig.8.5 crossing the stability limit $\delta_1$ destabilizes the cavity for CW operation while the cavity can still be stable for ML operation. The concept of re-stabilization is illustrated in Fig.8.10(a). The\[5\]The implication of this effect will be explained in Ch.10.2.
Figure 8.9: Kerr strength $\gamma$ as a function of the crystal position $Z$ at $EM2$ near the stability limit $\delta_2$. At $Z = 0$ the crystal center is located at a distance $f_2$ from $M2$. Positive values of $Z$ corresponds to translation of the crystal away from $M2$. Regardless of the crystal position, the sagittal plane has the higher value of $|\gamma|$.
ML mode which is sensitive to Kerr lensing is guided towards the long arm. Thus, the distance between $f_1$ and $f_2$ becomes effectively shorter for the ML mode compared to CW and the stability limit for mode locking is pushed towards higher values of $\delta$, while CW passively suffers from diffraction losses. Therefore, one can choose a mode locking point at $\delta > \delta_1$, which is stable for ML operation but not stable for CW operation.

The less effective hard aperture mode locking near $\delta_2$ can also be achieved passively without using a physical aperture by choosing the mode locking point at $\delta < \delta_2$. As illustrated in Fig.8.10(b), CW operation is destabilized while the cavity can still be stable for ML operation. The Kerr medium is pushed away from focus and the ML mode is imaged backwards. Thus, the distance between $f_1$ and $f_2$ becomes effectively larger for the ML mode compared to CW and the stability limit for mode locking is pushed towards lower values of $\delta$, while CW passively suffers from diffraction losses. Therefore, one can choose a mode locking point at $\delta < \delta_2$, which is stable for ML operation but not stable for CW operation. This however is much less efficient than mode locking at $\delta > \delta_1$.

Last, we note that hard/soft aperture mode locking can be achieved in the same manner as explained above near the stability limits $\delta_3$ and $\delta_0$. At $\delta_3$, hard aperture has the stronger nonlinear response, similar to $\delta_1$. However, due to experimental considerations, one usually prefers to have a collimated beam at one of the cavity arms, which is not the case at $\delta_3$ (see Fig.8.2). At $\delta_0$, soft aperture has the stronger nonlinear response, similar to $\delta_2$, yet cavity
8.3. Cavity analysis in mode-locked operation

Figure 8.10: Illustration of the Kerr lensing effect as a mode locking stabilizing mechanism: (a) increasing $\delta$ above the stability limit $\delta_1$, destabilizes CW operation while ML operation remains stable as the effective distance between $f_1$ and $f_2$ becomes shorter, (b) decreasing $\delta$ below the stability limit $\delta_2$, destabilizes CW operation while ML operation remains stable as the effective distance between $f_1$ and $f_2$ becomes larger.

operation near the plane-plane limit $\delta_0$ is usually less convenient compared to the point-plane limit $\delta_2$, since it is more sensitive to misalignment of the end mirrors.

The stabilization of the cavity for ML operation can also be explained by noting that the cavity will prefer to operate at the minimal threshold possible. For CW operation, the point where the threshold is minimal is close to the stability limits. This is because the threshold is proportional to the laser
Chapter 8. Cavity analysis

Figure 8.11: Illustration of the Kerr lensing effect as a mode locking stabilizing mechanism, in which the cavity seeks the minimal threshold possible.

mode size in the crystal and to the overlap between the laser mode and the pump mode. As can be seen from Fig. 8.11, the laser threshold is decreasing as $\delta$ approaches $\delta_1$ since the laser mode decreases in the crystal. The point of minimal threshold will be achieved when the laser mode on the crystal overlaps the pump mode. Beyond this point, threshold will increase again as the pump mode becomes larger than the laser mode. Threshold will continue to increase as the cavity is pushed into the non-stable zone. Therefore, pulsed operation is achieved when CW is destabilized and the laser seeks minimal threshold operation, by introducing the optimal Kerr lens that can restabilize the laser.
Chapter 9

Dispersion compensation

The ability of the optical cavity to sustain a pulse is highly dependent on the ability of the pulse to maintain its temporal duration as it circulates through the cavity. Since the cavity includes dispersive materials with a wavelength dependent refractive index, the optical cavity length is also wavelength dependent due to the fact that different wavelengths in the pulse travel at different speeds. This results in a wavelength dependent round trip time, which leads to temporal broadening of the pulse that limits the pulse duration and may even prevent mode locking. In the following, we provide a concise survey of the basic principles of dispersion management and the most commonly used dispersion compensation devices of prism pairs and chirped mirrors. For a more detailed study of the topic readers are referred to the literature [53, 54, 55].
9.1 Material dispersion

When a pulse is propagating through a dispersive material, its spectral phase \( \phi(\omega) \) is affected. By expanding \( \phi(\omega) \) into a Taylor series around the pulse central frequency \( \omega_0 \), one can identify three major effects, corresponding to the first three terms in the series\(^1\):

1. **Overall phase accumulation**, in which a constant phase is added to all frequencies.

2. **Group delay** (GD), in which the entire pulse is being delayed as a whole compared to a pulse propagating in free space.

3. **Group delay dispersion** (GDD), in which the blue components of the pulse are being delayed with respect to the red components\(^2\).

It is GDD that causes a temporal broadening of the pulse, hence it must be compensated to sustain the pulse over time\(^3\). For simplicity, higher order terms in the phase expansion, which correspond to third-, fourth- and fifth-order dispersion effects are neglected so far, yet the higher order terms are eventually the limiting factor for the shortening mechanism of the pulse duration. Expressions for higher order dispersion terms can be found in \[56\].

---

\(^1\)It is the non zero quadratic or higher order terms in the expansion that causes a pulse to be a non-transform limited (see also: Ch.3.2.2).

\(^2\)In this case, the material has normal dispersion, in contrast to anomalous dispersion.

\(^3\)GD is not a broadening effect.
9.1. Material dispersion

<table>
<thead>
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<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
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<td>1.42 $\cdot 10^{-2}$</td>
<td>3.25 $\cdot 10^2$</td>
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<tr>
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<td>0.23</td>
<td>1.01</td>
<td>6 $\cdot 10^{-3}$</td>
<td>2 $\cdot 10^{-2}$</td>
<td>1.03 $\cdot 10^2$</td>
</tr>
</tbody>
</table>

Table 9.1: Sellmeier coefficients for TiS crystal and BK7 glass. $C_i$ coefficients are given in units of $\mu m^2$.

It was shown that the temporal broadening of a pulse through a dispersive material with thickness $L$ is given by [57]:

$$\frac{\tau_{out}}{\tau_{in}} = \sqrt{1 + \frac{16(\ln 2)^2}{\tau_{in}} \left( \frac{d^2\phi}{d\omega^2} \right)_{\omega_0}}$$

(9.1)

where $\omega_0$ is the central frequency of the pulse and the phase is given by:

$$\phi(\omega) = (\omega/c)P(\omega),$$

where $P(\omega) = n(\omega) \cdot L$ is the optical path.

The refractive index $n$ is given by the Sellmeier equation:

$$n^2(\lambda) = \sum_i 1 + \frac{B_i\lambda^2}{\lambda^2 - C_i}$$

(9.2)

where $B_i$ and $C_i$ are experimentally determined coefficients for a given material.

The Sellmeier coefficients for the sapphire crystal and BK7 glass are given in Tab. 9.1.

Since the refractive index $n$ is given as a function of $\lambda$, it is convenient to calculate $d^2\phi/d\omega^2$ as a function of $\lambda$, given by:

$$GDD = \frac{d^2\phi}{d\omega^2} = \frac{\lambda^3}{2\pi c^2} \frac{d^2P(\lambda)}{d\lambda^2}$$

(9.3)

---

4The thickness $L$ is not excluded from $P(\omega)$ because in the general case the geometrical path $L$ can also be frequency dependent, as can be seen in Ch. 9.3.

5In this chapter, $\lambda$ is the wavelength in vacuum.

6Pulse broadening can also measured in terms of the group velocity dispersion (GVD) which is the GDD per millimeter of the corresponding material.
For a TiS crystal, we can approximate $L$ to be constant for all wavelengths. Since the crystal has normal dispersion within its emission spectrum, the result of Eq. 9.3 will be positive GDD for all wavelengths.

## 9.2 Chirped mirrors

In order to maintain near zero net GDD in the cavity, components of tuned negative GDD must be incorporated in the cavity. Chirped mirrors are a common component for dispersion compensation [58, 59] that are coated with a stack of dielectric layers, designed such that different wavelengths penetrate a different depth in the stack, as illustrated in Fig. 9.1(a). Mirrors are specified in the amount of negative GDD per bounce they provide. In many cases, a single mirror has a strong GDD oscillation across the spectrum, and pairs of mirrors are commonly designed with opposite oscillations, such that the combined GDD of the pair is spectrally smooth, as illustrated in Fig. 9.1(b). The technology of dielectric coatings for manipulation of ultrashort pulses has matured in recent years and now even double chirped mirrors are available for TiS cavities that provide specifically tuned negative GDD to compensate also higher order dispersion over an extremely wide spectral range, allowing to achieve pulses with extreme bandwidth (octave-spanning spectrum) [60, 61, 62]. Since compensation with chirped mirrors is discrete in nature (a finite dispersion per bounce on the mirror), one can use a wedge window pair to fine tune the dispersion. By controlling the insertion of one window, the variable thickness adds positive GDD in a controlled manner, allowing continuous compensation.
Figure 9.1: (a) Schematic diagram of a chirped mirror layer structure, (b) GDD values of a chirped mirror pair, in which the GDD oscillations in one mirror are compensated by the other, resulting in a constant negative GDD value for the entire spectrum.
9.3 Prisms-pair compressor

In contrast to material dispersion (where only the refractive index is wavelength dependent) one can also introduce geometric dispersion in which the geometrical path $L(\lambda)$ (Eq. 9.3) is also wavelength dependent. Geometric dispersion can be introduced using a prism pair, illustrated in Fig. 9.2, in either one of the cavity arms. The general concept of a prism pair is to manipulate the optical path of different frequencies in such a way that all the frequencies will experience the same cavity round-trip time. The relative time delay between the frequencies caused by the prism pair compensates for the time delay caused by other dispersive material in the cavity (e.g. the TiS crystal). The result is that a prism pair can generate both negative and positive GDD in a controlled, tuned manner [63, 64]. The GDD that the pulse experiences as it passes through the prism pair can be calculated using Eq. 9.3 where the optical path $P(\lambda)$ in the entire prism pair system (air and material) is given by:

$$P(\lambda, Z, h) = n_p(\lambda)d(\lambda, Z, h) + D(\lambda, Z, h) + \tilde{D}(\lambda, Z, h)$$ (9.4)

where $n_p$ is the prisms refractive index, $d$ is the length inside both prisms, $D$ is the length between the prisms and $\tilde{D}$ is the length after the second prism, measured with respect to an arbitrary reference point, e.g. the end mirror or any line parallel to the end mirror. For a given values of $Z$ and $h$, all these quantities can be geometrically calculated [65, 66].

---

7 The second derivation of $\phi(\omega)$ will be independent of this reference line.
8 A detailed explanation of how to measure $Z$ and $h$ for a given prism pair on the optical table is given in Ap. C.
9.4 Round trip phase accumulation

The net GDD that the pulse experiences as it passes through a prism pair can be continuously controlled, either by increasing the separation between the prism tips (adding negative GDD), or by increasing the prism penetration into the beam (adding positive GDD). A convenient configuration to increase positive GDD, is to fix the beam at the tip of the first prism, while translating only the second prism back and forth along the $h$-axis. Optimization guidelines for a prism pair setup can be found in [67], using a dispersion map in which the second and third order dispersion are plotted as orthogonal co-ordinates. The resulting dispersion vector can be compensated by optimizing the prism pair parameters (separation, penetration and prism material) to reduce the dispersion vector to zero. Similarly, a grating pair can be used for larger values of controlled negative dispersion [68] at the expense of increasing losses. The combination of a prism and a grating in a single device (termed: ’grism’) can also be used for higher order dispersion management [69]. One can also use a single prism and a wedged mirror to compensate for dispersion [70].

9.4 Round trip phase accumulation

After calculating the overall GDD that the pulse experiences through the entire cavity in a single round trip (material + prisms + mirrors), one can calculate the overall phase that the pulse accumulates in a single round trip by integrating the GDD function twice around the central frequency, as seen in Fig 9.3. The GDD is said to be compensated if the phase accumulated by the pulse is less than $2\pi/b$, where $b$ is the bounce number given by: $b = 1/Loss$, 

Figure 9.2: Geometry of a prism pair for dispersion compensation. This setup enables GDD manipulations in a controlled manner.

where \( Loss \) is the total losses in the cavity. This defines the bounce number as how many round trips does a photon experiences until it has a probability of \( 1/e \) to escape the cavity by transmission or loss mechanisms. The total losses in the cavity is given by:

\[
Loss = \sum_i (1 - \text{loss}_i)
\]  \hspace{1cm} (9.5)

where \( \text{loss}_i \) is the loss employed by a single cavity element. For example: in the cavity illustrated in Fig.8.6, the short arm end mirror \( EM1 \) has a reflectivity of \( R1 = 0.95 \) and the long arm end mirror \( EM2 \) has a reflectivity of \( R2 = 0.98 \). In addition, the TiS crystal has a thickness of \( L = 3\text{mm} \), absorption coefficient of \( \alpha_{\lambda=514} = 4.6\text{cm}^{-1} \) and figure of merit of \( FOM = 150 \). Hence, the total losses in the cavity is:

\[
Loss = (1 - R1) + (1 - R2) + \left( 1 - \exp \left( -\frac{\alpha_{\lambda=514}L}{FOM} \right) \right) \approx 0.08
\]  \hspace{1cm} (9.6)

which results in a bounce number of \( b \approx 12.5 \).
9.4. Round trip phase accumulation

Figure 9.3: Single round trip phase accumulation by a pulse in a cavity including: 3mm long TiS crystal, chirped mirror pair as plotted in Fig.9.1(b), prism pair of BK7 glass ($Z \approx 27cm$ and $h \approx 12cm$) and bounce number $b = 10$. 
Chapter 10

Additional cavity astigmatism

10.1 Prisms astigmatism

The prism pair is effectively a Brewster window separated in two parts. Therefore, the laser mode experiences astigmatism as it passes through the prism pair (as explained in Ch. 8.2.3). This astigmatic effect, which is generally much smaller than that of the crystal, is not included while solving Eq. 8.30 hence it will require fine tuning of the folding angles to recompensate the overall astigmatism in the cavity. Note however, that prism pair astigmatism is negligible if the beam in the prisms arm is collimated, i.e. the prisms are placed in the short arm while mode locking near $\delta_1$ or in the long arm while mode locking near $\delta_2$. This is because for a collimated beam, the prism pair astigmatism $\Delta L$ is much smaller than the Rayleigh range of the beam.

\footnote{Obviously, a convenient way will be to tune only the curved mirror that does not face the arm in which the prism pair are placed in.}
10.2 Nonlinear astigmatism

Kerr lensing in the Brewster cut TiS introduces an additional source of nonlinear astigmatism into the cavity. As the beam refracts into the crystal, the mode size in the tangential plane $w_t$ expands while the sagittal mode size $w_s$ remains the same, hence reducing the intensity and the nonlinear response in the tangential plane. The difference in the nonlinear response between the two planes will produce an intensity dependent astigmatism even if the linear astigmatism is fully compensated. In terms of the power dependent stability limit $\delta_1$ of mode locked operation illustrated in Fig.8.7(a), both the sagittal and tangential stability limits will be pushed towards higher values of $\delta$, but the sagittal limit will be pushed farther away compared to the tangential limit$^2$. In terms of the Kerr lens strength $\gamma$ plotted in Fig.8.8, both the tangential and sagittal planes will have a similar qualitative behaviour, but the absolute values of $\gamma$ will be reduced in the tangential plane compared to the sagittal. Therefore, a fully compensated CW mode will not remain compensated after mode locking. The standard solution to the problem is to pre-compensate the nonlinear astigmatism $^{[49]}$ by introducing extra linear astigmatism for the CW mode in the opposite direction, as seen in Fig.10.1, such that the plane with the stronger $|\gamma|$ will "catch up" with the weaker one. By changing the values of the angles $\theta_1$ and $\theta_2$ away from perfect linear astigmatism, one can pre-compensate for nonlinear astigmatism at $\delta_1$. Con-

$^2$Note that only hard aperture mode locking pushes the stability limits towards higher values of $\delta$, in contrast to soft aperture mode locking where the stability limits are pushed towards lower values of $\delta$. However, in both hard and soft aperture techniques, the relative variation of the power dependent stability limit will be higher in the sagittal plane compared to the tangential plane.
sequently, a deliberately non-circular CW mode will become circular after mode locking. Note however, that this compensation holds only for a specific value of $K = P/P_c$, i.e. specific intra-cavity peak power. Increasing (lowering) $K$ with the same folding angles (i.e. the same linear astigmatism) will result in over (under) compensating the nonlinear astigmatism. Any change in parameters that keeps the CW astigmatism compensated but affects the intra-cavity intensity, be it peak power or mode size in the crystal, will require a change in the folding angles to match the precise CW astigmatism needed to converge into a non-astigmatic ML beam. This includes a change in: pump power, pump focusing, output coupler, short arm length and also $Z$ or $\delta$. This requires specific compensation for every time one changes cavity parameters, making nonlinear astigmatism a major nuisance in standard cavity designs. We address this problem in Ch.13.
Figure 10.1: CW and ML astigmatism (defined as $\omega_s/\omega_t$) on the end mirror $EM1$ as a function of the crystal position for hard aperture mode locking at $\delta \lesssim \delta_1$. A linear astigmatism of $\approx 1.24$ is pre-introduced into the CW mode, allowing the stronger Kerr effect in the sagittal plane to "catch up" with the weaker tangential plane, resulting in astigmatically compensated ML mode at $Z \approx 0.08$. 
Chapter 11

Laser operation and optimization

11.1 Operation characteristics

The spectroscopic and laser characteristics in CW operation were well studied in the past and can be found in [39]. Pulsed operation of the TiS laser was also extensively studied, leading to remarkable results in terms of pulse duration [3], repetition rate [71], average power [72] and pulse energy [73]. Here, we provide some of the basic characteristics and typical qualitative behaviour of a TiS oscillator in ML operation, especially focused on the onset of ML and the experimental procedure to obtain ML. A top view picture of a typical TiS Kerr lens cavity is shown in Fig. 11.1. Using passive hard aperture mode locking (Ch. 8.3.3), pulsed operation can be achieved in a band of \( \delta \) values slightly beyond the stability limit \( \delta_1 \).

\[^{1}\text{This will be discussed in detail in Ch. 14}\]
Figure 11.1: Top view of a typical TiS cavity. The green, 532 nm pump beam is focused by $f_{\text{pump}}$ into the crystal. The red fluorescence can be seen on the crystal mount, along with the tubes circulating cold water to evaporate heat from the crystal.
Mode locking shows a qualitative behaviour, illustrated in Fig. 11.2(a). After crossing the CW threshold, pulsed operation is achieved only after the pump power is further raised to a certain ML threshold value, hence the appearance of pulsed operation is abrupt in terms of pump power. The threshold like behaviour of mode locking was elegantly explained, both theoretically and experimentally, as a first order phase transition by the theory of statistical light mode dynamics [74, 75].

Typical bandwidth of $> 200 \text{nm}$ at FWHM, as seen in Fig. 11.2(b), can be achieved even without the use of double chirped mirrors or additional bandwidth maximization techniques. Several techniques for temporal characterization of the pulse were developed in the past, such as: interferometric autocorrelation (IAC), frequency resolved optical gating (FROG) and spectral phase interferometry for direct electric-field reconstruction (SPIDER) [76, 77, 78, 79].

At the ML threshold, the output power of the pulse will be higher than the CW output power, since the pulse suffers from lower losses. Figure 11.3(a) plots the ratio between the pulse output power and the CW output power before mode locking as a function of $\delta$. The ratio $\gamma_e = P_{ML}/P_{CW}$ is an experimental measure of the Kerr lensing strength, similar to $\gamma$ of Eq. 8.38. In addition, the mode locking threshold as a function of $\delta$ is also plotted. We find that the ML threshold increases monotonically with increasing $\delta$, but the Kerr lensing effect has a maximum efficiency point. This behaviour will be discussed in detail in Ch. 14.

These techniques will be discussed in Ch. 11.2.
Figure 11.2: (a) Qualitative behaviour of pulsed operation for a given mode locking point at $\delta \gtrsim \delta_1$, (b) measured spectrum of a pulse with optimized dispersion compensation, resulting in a bandwidth of 205nm at FWHM. For a transform-limited pulse, this bandwidth corresponds to a temporal duration of $\approx 7.5\,fs$ at FWHM [80].
For a given mode locking point at $\delta \gtrsim \delta_1$, further increasing the pump above the ML threshold (Fig. 11.2(a)) will increase the CW output power, while the ML output power will remain approximately the same. This is because the pulse peak power is "quantized", i.e. there is a unique peak power that stabilizes the cavity for ML operation and the laser clutches to it. As the pump power is further increased, a small increase in ML power can be observed, but the CW power catches up quickly. When the CW output power becomes larger than the pulse output power, the excess energy excites a CW mode which oscillates in the cavity along with the pulse, hence the output spectrum will be a broad bandwidth with a narrow CW spike attached to it. The pulse peak power quantization is realized when the pump is further increased up to a point when there is sufficient energy to sustain two pulses in the cavity, as seen in Fig. 11.3(b). The onset of a double pulse is readily observed on the pulse spectrum as a fringe pattern due to spectral interference between the two pulses. Analysis of multi-pulse operation and single-pulse stabilization can be found in [81].

11.2 Optimization of laser parameters

So far, we described the operation concepts of the standard cavity design. The need to optimize the laser performance or to overcome inherent disadvantages and limitations motivated many extraordinary cavity designs. Since a comprehensive survey of all nonstandard cavity designs is beyond the scope, in the following we review a selection of published attempts to address these disadvantages and limitations.
11.2. Optimization of laser parameters

Figure 11.3: (a) ML threshold and ML-to-CW ratio of output powers as a function of \( \delta \), while mode locking using passive hard aperture at \( \delta > \delta_1 \). Measurements were taken from a typical configuration of TiS cavity including 3\( \text{mm} \) long Brewster-cut crystal, \( f_1 = f_2 = 5 \text{cm} \), \( L_1 = 20 \text{cm} \) and \( L_2 = 75 \text{cm} \), output coupler of 95\% reflectivity and the pump was focused to a diameter of \( \approx 22 \mu\text{m} \) in the TiS crystal, (b) measured spectrum of a single pulse with homogeneous spectrum and a double pulse with spectral interference fringes.
issues. In addition, the following review is a preface to chapters 12, 13 and 14 which also present novel cavity designs.

Various parameters of operation can be optimized using advanced cavity designs. Investigation of two- and three-mirror cavity configurations demonstrated highly compact cavities, in which the Kerr efficiency (Eq.8.38) was found to be maximal in a three mirror configuration. Optimization guidelines for the crystal length were given in [83] considering the estimated Rayleigh range of the laser mode in the crystal. Another novel cavity design was demonstrated in [84] by incorporating an acousto-optic Bragg cell at the end of the longer arm of the oscillator and using a curved mirror as an end mirror (instead of a flat mirror). The result is a fundamentally different diagram of the stability zones (compared to the conventional diagram shown in Fig.8.4) with a considerable improvement in the laser intensity stability.

Pulse duration can also be minimized by maximizing the spectral broadening of the pulse (by SPM, see Ch.5.2.2) inside the Ti:sapphire crystal. It was found that to maximize SPM, it is important to consider the physical arrangement of the elements in the cavity in order to achieve a symmetric dispersion distribution. When both cavity arms are compensated for dispersion independently [85], the nonlinear response can be twice stronger than the standard dispersion compensation, since it maximizes the nonlinear response in both the forward and backward propagation through the crystal. Thus, for maximal spectral broadening it is beneficial to consider either a four prism sequence in a ring cavity configuration, or to include a prism pair/chirped
mirrors in the short arm also. Spectral broadening can also be achieved using an additional Kerr medium at a second focus in the cavity [86].
Chapter 12

Intra-cavity gain shaping of mode-locked oscillations

In the following paper, demonstrate a method for precise control over the spectral gain in a mode locked oscillator in an intra-cavity, power preserving manner. By manipulating the gain properties in the cavity, we achieved almost complete control over the mode locked oscillation spectrum. The core principal of our method is to tailor the overall gain profile in the cavity, by combining the standard homogeneous gain with a small amount of spectrally shaped inhomogeneous gain, which allows us to selectively boost desired frequencies within the overall mode competition.

Beyond exploration into the basic physics of laser oscillation and mode locking, our experiments open a new avenue for shaping of ultrashort pulses, since it enables the realization of a dual-color ultrashort pulses inherently time synchronized and phase coherent. Even though the spectrum of pulses

http://www.opticsinfobase.org/oe/abstract.cfm?uri=oe-20-9-9991
may be easily shaped outside the optical cavity into a dual-color shape, this
shaping is inherently lossy, and for many applications impractical. Therefore,
many attempts were made to directly generate the required spectrum. Some
methods tried to shape the intra cavity loss with limited success, while others
make use of synchronization of several sources. Our method for flexible shap-
ing of the oscillation spectrum in a simple, power preserving manner within
one compact oscillator represents a major upgrade upon previous techniques
for ultra-fast sources and technology.

The dual-color ultrashort pulses generated by our source are inherently
time synchronized and phase coherent without any requirement for active
stabilization. Thus, these pulses are highly attractive for several very impor-
tant applications, such as Stimulated Raman Scattering (SRS) Microscopy or
Coherent Anti-Stokes Raman Spectroscopy\textsuperscript{2} (CARS). This method requires
picoseCONDS scale dual color sources that are synchronized with very low jitter,
and the lack of simple, robust sources was so far a major obstacle for
utilization of the considerable advantages offered by SRS microscopy. An-
other example is direct frequency comb techniques for precision spectroscopy
and coherent control of atomic and molecular dynamics\textsuperscript{3}. Those, as well
as other comb based precision measurements require concentration of comb
power at wanted spectral regions without compromising the phase stability
of the entire comb.

Intra-cavity gain shaping of mode-locked
Ti:Sapphire laser oscillations

Shai Yefet, Na’amam Amer, and Avi Pe’er*

Department of physics and BINA Center of nano-technology, Bar-Ilan university, Ramat-Gan 52900, Israel

*avi.peer@biu.ac.il

Abstract: The gain properties of an oscillator strongly affect its behavior. When the gain is homogeneous, different modes compete for gain resources in a ‘winner takes all’ manner, whereas with inhomogeneous gain, modes can coexist if they utilize different gain resources. We demonstrate precise control over the mode competition in a mode locked Ti:sapphire oscillator by manipulation and spectral shaping of the gain properties, thus steering the competition towards a desired, otherwise inaccessible, oscillation. Specifically, by adding a small amount of spectrally shaped inhomogeneous gain to the standard homogeneous gain oscillator, we selectively enhance a desired two-color oscillation, which is inherently unstable to mode competition and could not exist in a purely homogeneous gain oscillator. By tuning the parameters of the additional inhomogeneous gain we flexibly control the center wavelengths, relative intensities and widths of the two colors.

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OCIS codes: (140.3410) Laser resonators; (140.7090) Ultrafast lasers; (140.3538) Lasers, pulsed; (140.3580) Lasers, solid-state; (140.4050) Mode-locked lasers.

References and links
1. Introduction

Active oscillators, and in particular laser oscillators, which produce precise, stable oscillations are a major concept in physics and engineering. Due to the highly nonlinear interplay between positive feedback, loss, gain saturation (and possibly additional nonlinear effects), the steady state solution of a laser oscillation involves strong competition over gain resources between all possible modes of oscillation [1]. The ability to control mode competition and manipulate the oscillating cavity modes, stands at the heart of laser physics and applications.

The properties of the gain play a central role in the dynamics of a laser oscillator and in particular in determining the steady state oscillation. In textbooks of laser physics, gain is classified as either homogeneously or inhomogeneously broadened [2]. The effect of mode competition takes place in homogeneous broadening, where one global gain resource (population inversion in all the gain atoms) is accessible to all oscillation frequencies within the gain bandwidth of the active medium, causing the mode with the highest net gain to dominate over all the other possible modes. Consequently, in CW operation, a laser with homogeneous gain would inherently tend towards single mode operation. With inhomogeneous broadening however, each mode has its own gain resource and different frequencies in the oscillator do not compete. Consequently, a laser with inhomogeneously broadened gain would tend to multimode operation, with an oscillation spectrum that reflects the net gain spectrum. By shaping the loss profile one can enforce narrow oscillations on an inhomogeneously broadened laser at a cost of reduced pumping efficiency according to the ratio of the actual oscillation bandwidth to the bare gain bandwidth. For example, dual frequency oscillation can be obtained with inhomogeneously broadened gain by proper loss shaping and filtering only the desired frequencies (with an efficiency cost). With homogeneous gain however, only the identity of the single winning mode can be affected by loss shaping, but dual frequency oscillation cannot be enforced.

For a mode locked oscillation, an additional factor comes into play. First, the nonlinear loss imposed by the mode locking mechanism forces broadband, phase synchronized oscillations. Second, the spectrum is dictated by the delicate balance between gain, loss, dispersion profile and the temporal response of the mode locking mechanism. Due to these factors, the typical result is a pulse with a broad, smooth, single band spectrum. The situation with regard to intra-cavity spectral shaping however, is similar to CW. With inhomogeneous gain, multicolor operation is possible, as was demonstrated with mode locked semiconductor lasers near threshold [3], whereas for homogeneous gain, loss shaping can only set the allowed bands of oscillation, but between these bands, mode competition will usually choose one final winner.
oscillating band. Consequently, dual color modelocked oscillation is inherently unstable in a homogeneously broadened laser, and can be achieved only if the two colors have similar gain.

It so happens that most common mode locked lasers are primarily homogeneously broadened, mainly because the efficiency of pump utilization is higher for homogeneous gain. Even semiconductor lasers, which are inhomogeneously broadened near threshold, tend to become homogeneous as pumping is increased [4, 5]. Thus, if high power, dual color (or more) oscillation is desired, one must find a way to overcome mode competition in the homogeneous gain. We note that obtaining multi-color oscillation is more than an interesting exercise in laser physics; A multi-color (in particular dual color) oscillation is necessary for important applications, such as Raman spectroscopy [6], Raman microscopy [7, 8], and direct frequency comb spectroscopy [9, 10].

Many attempts were performed in the past to obtain dual color mode locking. For example, dual lobed loss filtering [11, 12] or dual output coupling [13] was attempted with minimal success, as these are inherently loss shaping techniques, that do not address the problem of mode competition. Other attempts bypassed the mode competition problem using active or passive synchronization of two independent lasers [14, 15], either by coupling two separate cavities through a shared gain medium [16, 17], or by synchronously pumping two OPOs [18]. All of these methods require several oscillators and special care for stabilization of timing jitter between the participating pulse trains.

Here we demonstrate a method to directly control the mode competition in a single oscillator, steering it towards the desired dual-color oscillation. The core principle is to tailor the gain profile instead of the loss inside the optical cavity. In addition to the homogeneous gain medium, we place a 2nd gain medium at a position in the cavity where the spectrum is spatially dispersed (as schematically shown in Fig. 1). In this position, different frequencies pass at different physical locations in the gain medium and therefore do not compete for gain. Furthermore, by proper spatial shaping of the pump beam in this additional gain medium one can shape the spectral gain profile. As opposed to the first homogeneously broadened gain medium, this additional gain is inherently inhomogeneous with a spectral shape of our desire. In this method, any combination of homogeneous and inhomogeneous gain can be realized by varying the splitting ratio of pump power between the two gain media. While most of the gain remains in the standard homogeneous medium, the additional inhomogeneous gain allows us to enhance specific frequencies in the overall spectrum by selectively adding gain to these frequencies, thus boosting them in the overall mode competition for the homogeneous gain. As demonstrated here on, this method allows steering the oscillation towards the desired double lobed (or more) spectrum, and shaping of the pulse spectrum almost at will, while preserving the total pulse power, and with minimal added pump power. We note that the concept of passively mode-locked laser with a single gain medium in the dispersive arm was introduced before [19]. However, using only inhomogeneous gain is highly inefficient in pump energy, as it requires pumping of a much larger volume. It is therefore much more efficient add only small amount of inhomogeneous gain in order to steer the competition in the homogeneous gain towards the desired oscillations.

Fig. 1. Block diagram illustrating the core principle of intra-cavity gain shaping.
Fig. 2. (a) Schematic of the standard design of a TiS oscillator. A linear cavity composed of a TiS crystal (Ti:Al₂O₃) as gain medium placed between two focusing curved mirrors (M1, M2), and a prism pair (or chirped mirrors) for dispersion compensation. (b) Schematic of the intra-cavity shaped oscillator. A 2nd TiS gain medium is placed at the Fourier plane of a 1×1 telescope placed between the prisms (Both TiS crystals were 3 mm long, 0.25 wt% doped). The telescope is comprised of two curved mirrors (M3, M4) of equal focal length \( f = 100 \text{ mm} \). Since the spectrum is spatially dispersed in the 2nd gain medium (each frequency component traverses at a different position), mode competition is canceled resulting in the ability to tailor the gain profile inside the oscillator by controlling the spatial shape of the pump in the 2nd gain medium. The inset shows a lateral view of the two pump spots in the 2nd gain medium.

2. Experimental and results

The standard design of a linear Ti:Sapphire (TiS) oscillator with a homogeneous gain medium is illustrated in Fig. 2(a), where a prism pair is commonly used to control dispersion [20]. Due to the homogeneous gain and without any loss shaping, the CW operation of the standard design is very narrow band, and the mode-locked operation is characterized by a pulse with a single band, broad, smooth spectrum. In our novel design, presented in Fig. 2(b), a unity magnification telescope is inserted between the prisms, with a second TiS crystal as an additional gain medium placed at the Fourier plane of the telescope. This effectively forms an intra-cavity pulse shaper, where it is possible to pump individual frequency components, and control the spectral amplitude of the oscillating modes.

The cancelation of mode competition in the 2nd medium is demonstrated in Fig. 3(a) by the continuous-wave (CW) operation of the novel cavity. The oscillator was pumped using a frequency-doubled Nd:YVO₄ laser at 532 nm (Verdi by Coherent) and when only the 2nd medium was pumped with an elliptically shaped pump spot we achieve a “multiple fingers” CW operation, indicating that different frequency components coexist. By controlling the shape of the pump spot for the additional gain medium, selected frequencies can be pumped simultaneously without mode competition. For a given prism material, these “fingers” span a bandwidth corresponding to the spatial width of the elliptically shaped pump spot, and can be centered
Fig. 3. Spectra of CW and pulsed operation of the cavity. (a) CW spectrum demonstrating cancelation of mode competition by the coexistence of multiple CW modes (fingers) when pumping only the 2nd medium with an elliptically shaped pump spot. In this study, the prisms was made of BK7 glass with dispersive power of $d\theta/d\lambda = 0.04 \text{ rad/\mu m}$, @ $\lambda = 0.8 \mu m$. Given a mode diameter of 21 \mu m the resolution of the intra-cavity shaper is 9.3 nm (the bandwidth occupied by a single mode on the surface of the 2nd gain medium). We used cylindrical optics to obtain an elliptically shaped pump beam of 21 \mu m \times 85 \mu m at the 2nd medium and we observed 4 CW fingers that span a bandwidth of \approx 35 nm, in good agreement with the expected 37 nm based on the above resolution. (b) Pulsed spectra observed in the cavity at different stages of pump transfer from the 1st medium (homogeneous gain) to the 2nd medium (spectrally selective gain). The 2nd medium is pumped at two selected frequencies with a tightly focused pump, resulting in a spectrum with two sharp lobes (red - initial, blue - intermediate, green - final spectrum).

anywhere within the TiS emission spectrum by scanning the pump beam laterally across the 2nd crystal. The number of modes (“fingers”) is determined by the ratio of the pump spot size to the resolution of the intra-cavity shaper.

The measured mode-locked operation is depicted in Fig. 3(b). First, the laser was mode-locked with only the 1st crystal pumped. Modelocked operation was achieved by standard techniques of mode-locking based on soft aperturing [21] and noise insertion (knocking on one of the prisms) [22]. Using pump power of 3.7W and an output coupler of 92%, we obtained an average modelocked power of 205mW with a broad homogeneous spectrum (\approx 140 nm @
FWHM, red curve) and a repetition rate of ≈ 100 MHz. Pump power was then transferred from
the 1st medium to the 2nd medium in several steps. In each step, we add small amounts of power
to the overall pumping, and then direct the excess pump power into the 2nd medium, where two
specified spots were pumped at positions corresponding to two lobes. During the transfer, gain
is increasing only for certain frequency components while decreasing for all other frequencies
(blue curve). The final shape of the spectrum has two clear and significant lobes (green curve),
one at 765 nm (20 nm @ FWHM) and the other at 840 nm (16 nm @ FWHM), and the inter-
mediate spectral power drops to < 10% from maximum. During the process of pump transfer
the average pulse power remained approximately the same (205 mW) and the dispersion profile
was tuned by translating the second prism in order to compensate dispersion for the desired two
colors. The spatial mode of the laser was stable and did not show any significant changes during
the entire process of pump transfer. Note that the control of the spectrum inherently requires
an increase in pump power, since one must pump (and cross threshold) in a larger volume of the
2nd medium, hence the overall pump power was increased during the process to maintain
pulsed operation up to a final level of 5.55 W which was split between the two media, such that
the pump power to the 1st medium dropped to 3.4 W, and 2.15 W of pump power power was
directed into the 2nd medium, further split between the lobes as follows: 1.3 W to the lobe at
840 nm and 0.85 W to the lobe centered at 765 nm. The splitting ratio is affected by the natural
gain as well as dispersion compensation at these wavelengths.

Since the transfer of the pump and hence the shaping of the pulse spectrum was a gradual,
adiabatic-like procedure, modelocking was preserved during the entire transfer from the initial
broad single band spectrum to the final two-color shape. This indicates that the dual-color
spectrum is inherently synchronized in time as a single pulse train, just like the original single
band spectrum was. There seem to exist however, an inherent limit on the amount of pump
power that can be transferred into the 2nd gain medium. After transferring about half the power
(and obtaining a well established two-color spectrum) any attempt to further transfer power
causes first the appearance of CW spikes and eventually brakes modelocking. In addition, even
when approaching this limit, the intermediate spectrum between the two forming lobes never
(and apparently cannot) drops to zero. The reasons for this limit are not fully understood, but it
seems as residual homogenous gain is necessary in order to maintain a broadband 'back bone'
to connect the two lobes, and to assure that the two colors are synchronized not only in time
but also in phase, forming one joint frequency comb, just like the initial unshaped pulse.

Figure 4 demonstrates the flexibility to control the spectral power, width and center wave-
length of each lobe. The green curve is that of Fig. 3(b), used here as a reference spectrum. By spatially widening the pump spots in the 2nd medium the spectral width of each lobe was
increased (blue curve). The spectral power of each lobe was controlled by adjusting the split-
ting ratio of the pump power between the spots in the 2nd medium (red curve). Shifting the
center position of the lobes by shifting the pump spot laterally is also demonstrated. The av-
average power is conserved for all curves at 205 mW, and the intermediate spectral power can be
reduced down to several percent only (< 3% from maximum).

A very important feature of our design is that once the spectrum profile is shaped as desired,
mode-locking directly into the shaped pulse is robust, without the need to repeat the step-by-
step pump transfer procedure. As seen in the Media 1 file attached to this paper, we could
repeatedly establish stable mode-locking directly into a narrow two-lobed spectrum with 90 nm
separation between the lobes and intermediate spectral power of 4% from maximum.

In our experiments, the places where lobes can be formed are limited by the spectrum of
the initial pulse with pure homogeneous gain before the pump transfer (Fig. 3(b), red curve).
Trying to pump frequencies beyond the FWHM bandwidth of the initial spectrum resulted in
the formation of CW spikes. A broader initial pulse can increase the bandwidth available for gain
Fig. 4. (Media 1) Control of spectral power, width and center position of spectral lobes. Taking a two lobed spectrum as a reference (green curve): control is demonstrated over the width of each lobe by changing the spatial width of the pump (blue curve) and the spectral power of each lobe by changing the power splitting ratio between lobes (red curve). The center of each lobe is also controlled by sweeping the pump spot position (left lobe shifted by 20 nm for both curves to 745 nm).

shaping ideally up to the entire emission spectrum of the TiS crystal. The minimum bandwidth of each lobe is limited by the spectral resolution of the intra-cavity shaper. Thus, using prism of higher dispersion, will reduce the bandwidth of the lobes for a given width of the pump beam, but at the same time will increase the pumping volume (and hence the pump power) needed for a given bandwidth of the lobes. The maximum width of the lobes depends only on the pump spatial profile and is limited by available pump power.

So far, the temporal shape of the two-lobed pulse has not been measured. We expect that the envelope of the pulse will have two characteristic time-scales: a short time-scale beating between the two lobes, contained within a long time-scale envelope of the entire pulse. The realization of a measurement system for such structured pulses with broad enough bandwidth on one hand, but with high enough spectral resolution on the other hand is not a simple task and is a future objective of this research.

3. Conclusion

We demonstrated a simple method to manipulate mode competition in a modelocked oscillator, based on a controlled combination of the standard homogeneous gain with a small amount of shaped inhomogeneous gain. This combination is very powerful for precise control over the spectrum of ultrashort pulses within the optical cavity, allowing stable oscillations that are inaccessible with purely homogeneous or inhomogeneous gain. Our concept of intra-cavity gain shaping holds notable advantages over other shaping techniques, either extra- or intra-cavity, as the former are lossy in power and the latter are very limited by effects of mode competition. Intra-cavity gain shaping provides, in a compact single oscillator, flexible, power preserving, so far unattainable control over the emitted pulses, and can generate multi-lobed spectra where the center, width and power of each lobe can be independently set.
Chapter 13

Kerr-lens mode locking without nonlinear astigmatism

In this paper\[1\] we discuss a novel cavity for KLM TiS lasers. The cavity configuration avoids the nonlinear astigmatism caused by the Brewster-cut TiS crystal and solves an inherent problem in the standard design of KLM cavities. The problem of nonlinear astigmatism is that it is power dependent, hence compensation for nonlinear astigmatism is power dependent as well. We overcome this problem by eliminating both the linear and nonlinear astigmatism of the TiS crystal, using a planar-cut crystal with plane parallel faces. A novel 3D cavity folding enables each of the curved mirrors to compensate for the astigmatism of the other. This eliminates the nonlinear astigmatism from the source, hence an astigmatically compensated CW beam will remain circular after ML regardless of the pulse peak power, as we show in this experiment.

\[1\]http://www.opticsinfobase.org/josab/abstract.cfm?uri=josab-30-3-549
Kerr lens mode locking without nonlinear astigmatism

Shai Yefet, Valery Jouravsky, and Avi Pe’er*

Department of Physics and BINA Center of Nanotechnology, Bar-Ilan University, Ramat-Gan 52900, Israel
*Corresponding author: Avi.Peer@biu.ac.il

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We demonstrate a Kerr lens mode-locked folded cavity using a planar (non-Brewster) Ti:sapphire crystal as a gain and Kerr medium, thus cancelling the nonlinear astigmatism caused by a Brewster cut Kerr medium. Our method uses a novel cavity folding in which the intracavity laser beam propagates in two perpendicular planes, such that the astigmatism of one mirror is compensated by the other mirror, enabling the introduction of an astigmatic free, planar-cut gain medium. We demonstrate that this configuration is inherently free of nonlinear astigmatism, which in standard cavity folding needs a special power-specific compensation. © 2013 Optical Society of America

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1. INTRODUCTION

Astigmatism is a well-known aberration in folded optical cavities that include Brewster-cut crystals and/or off-axis focusing elements [1]. For continuous-wave (CW) operation, astigmatism is linear and can be fully compensated by correctly choosing the folding angles of the focusing elements in accordance with the length of the Brewster-cut windows [2]. This results in a circular nonastigmatic beam at the output of the laser. For mode-locked (ML) operation induced by the nonlinear Kerr effect, an additional nonlinear astigmatism from the Kerr lens is added that is power dependent and needs to be taken into account. The standard technique for compensation for nonlinear astigmatism is to deliberately introduce linear astigmatism in the “opposite” direction [3], which results in compensation of the overall astigmatism for a specific intracavity intensity. Changing any of the cavity parameters that affect the intracavity intensity will require recompensation. Here we demonstrate a novel type of cavity folding that eliminates from the source nonlinear astigmatism in Kerr lens ML lasers.

2. STANDARD CAVITY DESIGN

Let us first review shortly the standard design of a ML Ti:sapphire cavity, illustrated in Fig. 1. The focusing mirrors are tilted with respect to the beam propagation axis, forming an X-fold (or Z-fold) cavity keeping the laser beam propagation parallel to the optical table. In this configuration, the astigmatism of the Brewster-cut crystal compensates for astigmatism of the curved mirrors.

The most important parameter for analyzing ML cavities is the strength of the Kerr effect defined as [4]:

\[
\gamma = \frac{P_e}{\omega} \left( \frac{d\omega}{dP} \right)_{P=0} ,
\]

where \(P\) is the intracavity peak power, \(\omega\) is the mode radius at the output coupler (OC) for a given distance between M1 and M2, and \(P_e\) is the critical power for catastrophic self-focusing [5]. From Eq. (1), it follows that the Kerr lens strength is represented by the change of the mode size on the OC due to a small increase in the intracavity peak power. This dependence of the mode size is due to the self-focusing effect caused by the intensity-dependent refractive index of the crystal: \(n = n_0 + n_2 I = n_0 + n_2 P/A\), where \(n_0\) and \(n_2\) are the zero-order and second-order refractive indices, respectively [6], and \(A\) is the mode area.

To determine the working point for ML, we define \(\delta\) as the distance between M1 and M2 with respect to an arbitrary reference point. Two separate bands of \(\delta\) values \([\delta_1, \delta_2], [\delta_3, \delta_4]\) allow stable CW operation of the cavity [7]. These two stability zones are bounded by four stability limits \((\delta_1 > \delta_2 > \delta_3 > \delta_4)\), each one requiring different angle values to compensate for linear astigmatism in CW operation. Astigmatic cavities are usually analyzed by splitting the cavity into tangential and sagittal planes, where both stable CW solution and \(\gamma\) are calculated for each plane separately. We note that while for CW operation, the uncoupled resonators calculation is quantitatively accurate, it is only qualitatively relevant for ML operation since the sagittal and tangential planes are coupled as the beam propagates through the Kerr medium [8] and the change of the beam size in one plane affects the lens strength also in the other plane. As we show hereon, this coupling is effectively nullled in the novel cavity folding, making the uncoupled resonators calculation quantitatively accurate.

It was shown in [9] that \(|\gamma|\) is maximized close to the cavity stability limits. We therefore calculated \(|\gamma|\) using the transformed complex beam parameter method presented in [10] near the second stability limit \(\delta_2\) at a position corresponding to a beam size with a typical diameter of 2.45 mm. Calculations are performed within the aberration-free Kerr lens approximation, in which the transverse variation of the refractive index is approximated to be parabolic, so that the beam maintains its Gaussian shape during the propagation and ABCD matrices to analyze the cavity can be applied.

Figure 2 plots the normalized \(|\gamma|\) of the sagittal and tangential planes for the standard cavity folding as a function of the crystal position Z. The folding angles were chosen so that linear CW astigmatism is compensated, resulting in a circular CW beam on the OC. As the beam refracts into the crystal at Brewster angle, the mode size in the tangential plane increases by a factor of \(\eta_0 = 1.76\), whereas in the sagittal plane does not change. Since
3. NOVEL CAVITY DESIGN

Here we demonstrate a different type of cavity folding which allows the introduction of a planar-cut (non-Brewster) crystal where the laser beam enters the crystal at normal incidence. In this configuration, the curved mirrors compensate for the astigmatism of each other instead of the crystal, and the spatial mode of the beam in both planes does not change as it enters the crystal, thus cancelling the nonlinear Kerr lens astigmatism from the source. Figure 3 illustrates the laser configuration. Mirror M1 folds the beam in-plane of the optical table, whereas M2 folds the beam upward. Thus, the sagittal and tangential components of M1 exchange roles at M2, leading to exact cancellation of the linear astigmatism of one mirror by the other mirror. Note that the polarization of the laser is unaffected by the new folding, which interchanges the primary axes but does not mix them. The normalized Kerr lens strength for this configuration is plotted in Fig. 4 for the same δ as in the standard folding (other cavity parameters remain unchanged). It is clear that the Kerr lens strength of each plane is equal with a small separation between the curves of each plane due to the nonzero value of the angles, taken to be θ = 2.5°, the minimal value possible with the optomechanical mounts in our experiment.

We note that in this configuration, the mirrors’ astigmatism is uncoupled from the astigmatism of the Brewster-cut windows. In order to completely compensate for linear astigmatism, the slight astigmatism from the Brewster-cut prism pair must be taken into account. Thus, equal folding angles will compensate for astigmatism only when the beam is collimated in both arms (at δ1). For δ2 however, equal angles can still be maintained using the prisms’ astigmatism to compensate for the residual mirrors’ astigmatism at δ2 by correctly choosing the overall propagation length inside the prisms.
Experimentally, it was critical to slightly tilt the crystal so that reflections from its surface will not interrupt the ML process. We measured that for ML to operate, the minimal incidence angle of the laser beam at the crystal surface was $\approx 3.7^\circ$, which causes a slight increase in the tangential mode size inside the crystal, thus resulting in a negligibly small nonlinear astigmatism. To estimate the nonlinear astigmatic effect due to the crystal tilting angle, we compare the decrease in the tangential optical path in the crystal with respect to that of the sagittal being $L/n_0$. Using ABCD matrices \([11]\), the decrease factor in the tangential plane for Brewster angle can be calculated to be $1/n_0^2 \approx 0.32$, while for $3.7^\circ$, this factor becomes 0.997 and can be completely neglected.

4. RESULTS

The cavity was mode locked using pump power of 4.6 W focused to a diameter of 45 $\mu$m in the gain medium (5 mm long, 0.25 wt. % doped) with an OC of 85%. A prism pair of BK7 glass with 40 cm separation was used for dispersion compensation. The cavity produced $480 \text{ mW}$ of pulse output power with spectral bandwidth of $\approx 100 \text{ nm}$. We note that the pulse bandwidth is not fully optimized and broader bandwidths can be achieved with better dispersion compensation.

Figure 5 plots the intensity profile of the CW and ML beams in the near field and for the ML beam also in the far field as measured by a CCD camera. For the near field, the CW astigmatism (defined as the ratio between the sagittal ($\omega_s$) and tangential ($\omega_t$) components) was fitted to be 0.98, in good agreement with our prediction, and the beam has an average radius of $0.5(\omega_s + \omega_t) = 0.66 \text{ mm}$. As expected, the ML astigmatism remains exactly the same with an average beam radius of 1.38 mm. The focusing quality of the ML mode was measured in the far field, showing a beam quality factor of $M^2 = 1.6$.

We note that using a planar-cut crystal reduces the mode size inside the crystal compared to a Brewster-cut crystal, since the beam does not expand in one dimension upon refraction into the crystal. This can reduce the laser threshold and increase the pulse output power compared to a standard cavity folding of similar parameters. In addition, the reduced mode size enhances the nonlinear Kerr lensing compared to Brewster-cut crystals, leading to an increased mode-locking strength. The large difference in beam size between the ML and CW modes (Fig. 5) is a result of the improved Kerr strength, demonstrating the higher efficiency of the Kerr nonlinear mechanism in this design.

5. CONCLUSION

To conclude, we demonstrated a cavity design that in the first place needs no compensation for nonlinear astigmatism. In addition, the range of the crystal position in which both sagittal and tangential components contribute to the ML process is increased, resulting in a simple robust configuration for ML lasers.

ACKNOWLEDGMENTS

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Chapter 14

Mode locking at and below the CW threshold

In this paper\footnote{http://www.opticsinfobase.org/oe/abstract.cfm?uri=oe-21-16-19040} we demonstrate a mode-locked Ti:Sapphire laser with an enhanced Kerr lensing mechanism, operating in a new regime, where the pump power needed to sustain a pulse is below the pump power needed to start CW oscillation. In this regime, the mode locking procedure can be started directly from zero intra-cavity power. In this regime the nonlinear Kerr effect is truly and fully exploited, allowing pulse formation even when there is no stable solution for CW operation. We experimentally explore this regime and provide a theoretical model that describes the dynamics and formation of pulses in this regime. We also find that with our laser, pulse parameters are improved compared to record results, in terms of low intra-cavity power and decreased pump threshold.
Mode locking with enhanced nonlinearity - a detailed study

Shai Yefet and Avi Pe’er

Department of physics and BINA Center of nano-technology, Bar-Ilan university, Ramat-Gan 52900, Israel

*avi.peer@biu.ac.il

Abstract: We explore mode locked operation of a Ti:Sapphire laser with enhanced Kerr nonlinearity, where the threshold for pulsed operation can be continuously tuned down to the threshold for continuous-wave (CW) operation, and even below it. At the point of equality, even though a CW solution does not exist, pulsed oscillation can be realized directly from zero CW oscillation. We experimentally investigate the evolution of the mode locking mechanism towards this point and beyond it, and provide a qualitative theoretical model to explain the results.

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References and links
1. Introduction

The ultra-broad gain bandwidth of the Ti:Sapphire (TiS) laser renders it the ‘work-horse’ of the last decades for generation of ultrashort pulses by mode locking (ML) [1]. The nonlinear mechanism responsible for ML is self-focusing of the beam due to the optical Kerr effect within the TiS crystal, introducing an intensity dependent loss mechanism that favors pulses over continuous-wave (CW) operation [2]. A well known feature of ML is the abrupt transition between CW and ML operation in terms of pump power [3]. Only when the pump power crosses a certain threshold, can ML be initiated from a noise-seed (either by a knock on a cavity element or by external injection of long pulses). Another common feature is that the threshold pump power for ML is higher than the CW threshold. Typical mode locked operation requires a certain amount of CW oscillations to exist in the cavity, and only on top of the existing CW can an intensity fluctuation be amplified to create the pulse. In the following, we investigate the possibility to push the ML threshold to the extreme, i.e. the possibility of reducing the ML threshold down to the CW threshold and even lower.

The key factor for reducing the ML threshold is enhanced intracavity nonlinearity, as elegantly explained by the theory of statistical light-mode dynamics (SLD) [4]. In SLD, the mode locking procedure is transformed into a problem in statistical mechanics and the transition from CW to pulsed operation is described as a first order phase transition, which explains the threshold-like behavior of mode locking. The order parameter (analogous to temperature) is $T = N / (\gamma_s P^2)$, where $N$ is the noise level in the cavity, $\gamma_s$ represents the strength of the relevant nonlinearity and $P$ is the total laser cavity power, that can be controlled by pump power. It was theoretically calculated that mode locking occurs below a critical value $T_c$ of the order parameter, and it was verified experimentally in a mode locked fiber laser [5] that as the pump power increases, pulsed operation can be initiated with higher values of noise injected to the laser (for a constant nonlinear strength $\gamma_s$). Therefore, an increase in the intracavity nonlinearity will be accompanied by a decrease in the mode locking threshold.

Cavities with enhanced intracavity nonlinearity were explored in the past [6–12], showing an overall reduction of the mode locking threshold, but the limits of this phenomenon were not directly explored. In this work, we perform a detailed survey, where the Kerr nonlinearity is tuned continuously, in order to investigate the ML threshold dependence on the intracavity nonlinearity. We monitor the mode locking strength and threshold across a large parameter range and observe the optimal intracavity power for mode locking and its dependence on nonlinearity. At this optimal point, we show that with sufficiently enhanced nonlinearity, the intracavity power necessary to initiate pulsed operation can be reduced down to zero. We note that by “zero” we mean the first appearance of small CW fluctuations that experimentally identifies the threshold point, which is always smeared by noise and never completely defined.
Fig. 1. Cavity configuration. The gain medium is a 3 mm long Brewster-cut TiS crystal with 0.25 wt% doping. The curved mirrors (M1,M2) radius of curvature is \( R = 15 \text{cm} \), with high reflector (HR) and a 95% output coupler (OC) as end mirrors. An additional planar-cut BK7 window is inserted near the image point of the TiS crystal, created by the two-lens telescope of focal length \( f = 10 \text{cm} \). The short cavity arm is 42 cm long and the long arm (90 cm) contains a prism-pair of BK7 glass (60 cm). Each cavity mirror except the OC provides group velocity dispersion of \( GV D \approx -55 \text{fs}^2 \). The oscillator is running in the regime of anomalous dispersion.

2. Experimental setup

Our experimental setup is a standard X-fold cavity with a prism pair for dispersion compensation, as illustrated in Fig. 1. In order to enhance the intracavity nonlinearity in a controlled manner, we introduce a lens based 1x1 telescope between the curved mirrors. The focus inside the TiS crystal is thus imaged to a distance \( 4f \) towards mirror \( M_1 \), where we can insert an additional planar cut Kerr medium near the imaged focus. The choice of a lens based telescope and a planar cut Kerr window (instead of the typical configuration [6] of a mirror based telescope and a Brewster cut window) is motivated by the enhanced nonlinearity and lack of nonlinear astigmatism offered by this configuration. In a Brewster cut Kerr medium the beam expands in one dimension (due to refraction) and hence the nonlinearity of the material is reduced and becomes astigmatic [13]. A planar cut Kerr medium does not suffer from beam expansion and hence the Kerr effect can be fully exploited and astigmatism free. The lens based 1x1 telescope is also astigmatism free, providing a simple expansion of the standard TiS configuration without changes of angles/mirrors (expect for some additional dispersion).

If we define \( \delta \) as a measure for the distance between \( M_1 \) and \( M_2 \) with respect to an arbitrary reference point, we find two separate ranges of \( \delta \in \Delta \), which allow stable CW operation of the cavity, bounded by four stability limits (\( \delta_1 > \delta_2 > \delta_3 > \delta_4 \)). The working point for ML in our experiment is the typical working point, near the second stability limit \( \delta_2 \) [2]. Near this limit, the nonlinear Kerr lensing effect causes a decrease of the mode size at the output coupler for pulsed operation compared to the CW mode size. In order to favor ML over CW operation, one can use the hard aperture technique, which can be implemented in two ways. The first is to close an aperture near the OC for a fixed value of \( \delta \). This actively induces loss on the larger CW mode, while for ML, the Kerr effect reduces the mode size, allowing the beam to pass through the aperture. Further closing the aperture will enforce pulses with higher intracavity power. The second method is to increase the distance \( \delta \) between the curved mirrors beyond the stability limits, without using a physical aperture. This passively induces increasing diffraction losses on the CW mode, while the additional nonlinear Kerr lens eliminates them and stabilizes the cavity for ML. In this manner increasing the distance \( \delta \) is equivalent to closing a
Fig. 2. ML (red) and CW (blue) operation parameters as a function of $Z = \delta - \delta_2$ for two $3\text{mm}$ long BK7 window positions: off-focus (a)+(b) and in-focus (c)+(d). Mode locking at the critical point $Z_c$ where the ML threshold equals the CW threshold is demonstrated in the attached media file (Media 1).

physical aperture on the beam. Note that the absence of a physical aperture does not necessarily leads to the soft aperture mode locking technique. In soft aperture, a physical aperture cannot be used at the short arm, let alone to increase the curved mirror separation.

3. Results

We introduced as an additional Kerr medium a $3\text{mm}$ long planar cut window of BK7 glass. Figure 2 plots the measured ML and CW operation parameters as a function of $Z = \delta - \delta_2$. Measurements were taken for two positions of the BK7 window: 1. at the imaged focus, where considerable nonlinearity is added by the window (in-focus), 2. several centimeters away, much beyond the Rayleigh range of the intracavity mode (off-focus), where the added nonlinearity is negligible and the cavity acts as a standard Ti:S cavity (with some additional material dispersion). Figure 2(a) plots the CW threshold and the ML threshold as a function of $Z$ for off-focus position. The ML threshold is defined as the minimum pump power required to initiate pulsed operation. As expected, the CW threshold increases with $Z$, due to increased diffraction losses (similar to closing an aperture). The ML threshold also increases (since higher power is needed for ML to overcome the loss). Figure 2(b) shows the CW and ML intracavity powers at the ML threshold as a function of $Z$ for off-focus position. From both figures we see the typical behaviour of mode locked lasers, where the ML threshold is always larger than the CW threshold and CW oscillation must exist to initiate the ML process. In addition, although ML operation is favorable over the entire range of $Z$, it is most favorable at the "sweet spot" ($Z_{ss} \approx 1.1\text{mm}$), where the CW oscillation required to start the ML process reaches a minimal value, due to maximum self amplitude modulation.

The same CW and ML parameters for in-focus position are plotted in Figs. 2(c) and 2(d). The overall absolute values of the ML threshold are reduced by the added nonlinearity, and the ML threshold curve eventually crosses the CW threshold at $Z_c \approx 1.2\text{mm}$ where the intracavity CW power drops to zero. At $Z_c$, ML can be achieved directly from the CW threshold. The
corresponding CW and ML intracavity powers at the ML threshold are shown in Fig. 2(d), resulting in a reduction of the ML intra-cavity power as the intra-cavity nonlinearity increases. The attached Media 1 file demonstrates pulsed operation at the critical point $Z_c$. In the first part (spectrum), there is no signal in CW. By knocking on one of the end mirrors (we do not observe self-starting operation), an instantaneous and extremely weak intensity spike is generated right at the CW threshold and immediately evolves into a pulse. In the second part (power), there is a $\mu W$ signal (from crystal fluorescence), and the power jumps to $\approx 80mW$ when ML. In the third part (spatial mode), a clear circular ML mode appears on an IR card without preliminary CW mode. To verify the procedure reproducibility, the pulse is deliberately broken and re-modelocked.

Beyond the crossing point ($Z > Z_c$), ML can still be initiated, but only by first raising the pump power up to the CW threshold, locking, and then lowering the pump again. In this regime, at the CW threshold the pump power is already too high and mode locking generates a pulse with a CW spike attached to it, which can be eliminated by lowering the pump power below the CW threshold. The ML threshold in Fig. 2(c) for $Z > Z_c$ is the minimal pump power needed to maintain a clean pulse.

We can understand the need to first increase the pump power to the CW threshold and only then lower it, by noting that the CW threshold marks the crossover between decay and amplification in the cavity. For ML to occur, an intensity fluctuation must first be linearly amplified to a sufficient peak power to initiate the Kerr-lensing mechanism. For $Z > Z_c$, one must pump the laser sufficiently for a noise-induced fluctuation to be amplified (rather than decay) in order for it to reach the peak intensity required to mobilize the Kerr-lensing process. After reducing the pump power to the ML threshold, a clean pulse operation is obtained, but if ML is broken the cavity will not mode-lock again, as it does not even lase.

It is important at this point, to differentiate between the laser behaviour at $Z > Z_c$ and hysteresis effects, which are common and typical to mode locking. The hysteresis in mode locked lasers is identical to the supercooling effect in liquids. Mode locking hysteresis is the need to raise the pump power more than what is needed to sustain a pulse, in order for the laser to leave its meta-stable state and to initiate pulsed operation. Once the pulse is generated the pump power can be reduced to a lower value, but if the pulse is broken, the cavity will not mode lock again. The difference between the minimal pump power needed to initiate the pulse and the minimal pump power needed to sustain the pulse is the hysteresis gap. Hysteresis, therefore, can be observed only at $Z < Z_c$ (and was not experimentally observed in our setup). This is not the case at $Z > Z_c$, since raising the pump to the CW threshold is not because of hysteresis, but it is a necessary condition for the laser to be an amplifier at all. Below the CW threshold, the system acts as an attenuator, not an amplifier, and a meta-stable state cannot exist in the first place. Hence, this effect is inherently different from the commonly observed hysteresis loop. In this regard, our work is different from [14] where a considerable reduction of the ML threshold was achieved using a gain matched output-coupler, showing a minimum, but non-zero, CW power can be reduced to a lower value, but if the pulse is broken, the cavity will not mode lock again. The difference between the minimal pump power needed to initiate the pulse and the minimal pump power needed to sustain the pulse is the hysteresis gap. Hysteresis, therefore, can be observed only at $Z < Z_c$ (and was not experimentally observed in our setup). This is not the case at $Z > Z_c$, since raising the pump to the CW threshold is not because of hysteresis, but it is a necessary condition for the laser to be an amplifier at all. Below the CW threshold, the system acts as an attenuator, not an amplifier, and a meta-stable state cannot exist in the first place. Hence, this effect is inherently different from the commonly observed hysteresis loop. In this regard, our work is different from [14] where a considerable reduction of the ML threshold was achieved using a gain matched output-coupler, showing a minimum, but non-zero, CW power could be reduced even further, below the CW threshold in a hysteresis like effect, as predicted by [15].

To investigate the appearance of the regime $Z > Z_c$ for in-focus window position, we plot in Fig. 3(a)-(d) the ratio of CW to ML powers $\gamma \equiv P_{CW}/P_{ML}$ as a function of $Z$, for windows of variable thickness. $\gamma$ represents an experimental measure for the strength of the Kerr effect, which demonstrates a "sweet spot" $Z_{cs}$ where $\gamma$ is minimal and the nonlinear mechanism is most efficient. The apparent tendency from Fig. 3 is that for increased nonlinearity, the $\gamma$ value at the sweet spot is reduced and the sweet spot is pushed to larger values of $Z$. We find that for a 2mm thick window the $\gamma$ curve touches on zero near the sweet spot, marking the onset of the new regime, where mode locking can be initiated from the CW threshold resulting
Fig. 3. (a)-(d) Experimental definition of the Kerr strength as a function of $Z$ for BK7 window with different lengths, (e) measured spectrum for 2mm BK7 window at $Z_{SS}$.

in a pulse with energy of $\approx 30\text{nJ}$. The pulse spectrum is plotted in Fig. 3(e), resulting in a bandwidth of $\approx 140\text{nm}$ at FWHM, which corresponds to a transform-limited pulse duration of $\approx 12\text{fs}$ at FWHM. For a 3mm thick window the curve crosses zero at $Z = Z_c$ and the sweet-spot location $Z_{SS}$ cannot be directly measured. Well above $Z > Z_c$ pulsed operation becomes unstable, and we could not observe the reappearance of the $\gamma_e$ curve for larger values of $Z$. At every experimental point the prisms were adjusted to provide the broadest pulse bandwidth. The maximum oscillation bandwidth was obtained near the sweet spot due to the maximized Kerr strength, reaching approximately the same bandwidth for all of the window thicknesses used. This indicates that the bandwidth was limited mainly by high order dispersion of the prisms-mirrors combination, and not by the added dispersion of the windows. In addition, we havent seen any significant change in the mode locking starting mechanism, and pulsed operation was always started by a gentle knock on the end mirror.

4. Discussion

To provide a qualitative model for the dynamics of the "sweet spot" with increasing Kerr non-linearity we examine a commonly used theoretical measure for the Kerr Strength:

$$\gamma_s \equiv \frac{P_c}{P} \frac{d\omega}{dP}$$

where $\omega$ is the mode radius at the output coupler and $P$ is the pulse peak power normalized to the critical power for self-focusing $P_c$ [16]. This Kerr strength, which represents the change of the mode size due to a small increase in the ML power, is a convenient measure for mode-locking with an aperture near the output coupler. Yet, since increasing $Z$ is equivalent to closing an aperture, $\gamma_s$ is useful also for our configuration. Usually, $\gamma_s$ is calculated at zero power ($P = 0$) [17] to estimate the tendency of small fluctuations to develop into pulses, however, the dependence of $\gamma_s(P)$ on power is also important. Specifically, a large (negative) value for $\gamma_s$ indicates that only a small increase in the ML power (or threshold) is necessary to overcome a reduction of the aperture size (or increase in $Z$). The power where $\gamma_s$ is most negative, represents therefore a sweet spot for mode locking. $\gamma_s$ is calculated using ABCD matrices [18] and the transformed complex beam parameter [19].

Figure 4 plots $\gamma_s(P)$ for no added Kerr window (TiS only) and for a 2mm long added window, demonstrating a clear minimum (sweet spot) on both curves. Furthermore, as Kerr material is added (enhanced Kerr strength), the minimum point is deepened and pushed towards higher power values(larger $Z$), similar to the observed $\gamma_e$. Although $\gamma_s$ and $\gamma_e$ are somewhat differ-
ent measures for the Kerr strength, the calculation of $\gamma$ provides reasoning for the measured behavior of the sweet spot with the different window thicknesses.

The above study of the ML threshold for varying Kerr strength provides insight to the mode locking dynamics, and an experimental verification of the critical point where ML can be initiated from zero CW power. The ML laser reported here at the critical distance $Z_c$, can be compared to recently published results [20] that reported a low threshold ML TiS laser with an output coupler of 99%, output power of $30mW$ and intracavity power of $3W$. Here, we have achieved ML from zero CW oscillation with similar repetition rate ($\approx 80MHz$), at much lower intracavity power and less stringent conditions. In our experiment, the output coupler had only 95% reflectivity (5 times more losses), coupling more power out ($\approx 85mW$) and considerably lower intracavity power of $1.7W$. With further optimization, the enhancement of the cavity non-linearity may allow development of ML sources with ultra-low intra-cavity power. More importantly, our method may allow mode locked operation with high repetition rates ($>1GHz$), which is desired for frequency comb applications. As the repetition rate is increased, the pulse energy is reduced, reaching eventually the point where pulsed operation can no longer be sustained. Increasing the intra-cavity non-linearity can compensate for the reduction in pulse energy in lasers with high repetition rate.

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Chapter 15

Concluding Remarks

15.1 Summary

To conclude, novel designs of a Kerr-lens mode-locked Titanium-sapphire laser were developed, solving inherent disadvantages and limitations relevant to the standard cavity designs:

1. In the first paper, we have demonstrated precise control over the spectral gain in the oscillator. We have achieved almost complete control over the mode locked oscillation spectrum, while controlling this spectrum inside the cavity itself in a power preserving manner, in contrast to lossy spectral manipulations outside the oscillator. In addition, the final two lobes of the pulse are synchronized, because the spectral shaping of the inherently, fully synchronized initial pulse is done in an adiabatic, step-by-step procedure, while maintaining mode locking throughout the entire pump transfer. Our technique solves the prob-
lem of mode competition, hence the synchronization of several sources is not necessary.

2. In the second paper, we have demonstrated a novel configuration the laser cavity, in which the nonlinear astigmatism in completely cancelled and a planar-cut TiS crystal can be introduces instead of the standard, Brewster-cut crystal. The unique folding of the laser enables one to compensate only for linear astigmatism in the cavity so that nonlinear, intensity dependent, optical aberrations does not exist, hence an astigmatically compensated CW beam will remain circular after mode locking regardless of the pulse peak power. We note that indeed, a Brewster-cut TiS crystal has an advantage upon AR coated, planar-cut crystal in terms of price. However, if we take into account the price of the entire cavity elements and the pump system, the AR coating is a negligible addition with respect to the overall price.

3. In the third paper, we have demonstrated a new regime of operation of the laser, in which the nonlinear Kerr effect is truly and fully exploited. In this regime, the pump power needed to sustain a pulse is below the pump power needed to start CW oscillation and the mode locking procedure can be started directly from zero intra-cavity power. In addition, laser operation properties were improved and a new insight on the physics of Kerr-lens mode locking was revealed.

15.2 Prospects of further study

Further investigation can be made in any one of the above detailed works:
15.2. Prospects of further study

1. **Frequency comb stabilization:** in order for a pulse to act as a probing devise for high resolution spectroscopy, it has to be stabilized in order to become a frequency comb\(^1\). As explained in Ch.3.2.3, the two degrees of freedom of a frequency comb, the repetition rate \(f_{\text{rep}}\) and the offset frequency \(f_0\), must be stabilized. The stabilization procedure will be much interesting for the two-color comb source presented in Ch.12.

2. **Temporal profile measurement:** the temporal profile of the two-color pulse was not directly measured yet, mainly because precise measurements of such structured pulses is not simple. We expect that the envelope of the pulse will have two characteristic time-scales, demonstrating a fast modulation due to the beat between the lobes contained within a slow envelope due to the spectral width of each lobe. The realization of a FROG to characterize such pulses with broad enough acceptance bandwidth, yet with high enough spectral resolution, is not a simple task. Therefore, setting up a suitable measurement system to address questions concerning the temporal domain is also a good candidate for a future study. Nevertheless, a numerical calculation simulating the evolution of pulses in an optical cavity with spectrally controlled gain was carried out, showing preliminary results of the pulse temporal structure in accordance with our predictions.

3. **Ultra-low mode locking threshold:** the cavity designs presented in Ch.13 and Ch.14 can be combined into a single optical cavity. Every one of the two cavity configurations in these chapters was found to act as a

\(^1\)See Ch.3.2.3
lowering mechanism of the laser threshold. A combined configuration will be a good candidate to realize record results for the threshold of KLM TiS lasers.
Appendix A

Dioptric power of a nonlinear Kerr-lens

In the following, we derive the expression of the dioptric power of a thin slice of Kerr material, given by Eq.5.2. Here, we derive the general expression for a non-circular Gaussian beam with $\omega_x \neq \omega_y$.

The phase accumulated after passing thorough a material with nonlinear refractive index given by Eq.5.1 and thickness $z$ is given by:

\[
\phi = kz \cdot n(I) = kz(n_0 + n_2I) = kzn_0 + kzn_2I \quad (A.1)
\]

We can neglect the constant phase which does not depend on $I$ and substitute a Gaussian intensity profile:
\[
\phi = k zn_2 I_0 e^{-2 \left( \frac{x^2}{\omega_x^2} + \frac{y^2}{\omega_y^2} \right)} \\
= k zn_2 \frac{P}{\pi \omega_x \omega_y} e^{-2 \left( \frac{x^2}{\omega_x^2} + \frac{y^2}{\omega_y^2} \right)} \tag{A.2}
\]

We can expand \( e^{f(x,y)} \approx 1 + f(x,y) \), neglecting again the constant phase:

\[
\phi = -\frac{4zn_2 P k}{\pi \omega_x \omega_y} \frac{x^2}{\omega_x^2} + \frac{y^2}{\omega_y^2} \tag{A.3}
\]

Equation (A.3) can be compared to the phase accumulated by a simple lens, given by:

\[
\phi_{\text{lens}} = -\frac{k}{2} \left( \frac{x^2}{f_x} + \frac{y^2}{f_y} \right) \tag{A.4}
\]

Therefore, the Kerr lensing for each plane is given by:

\[
\frac{1}{f_x} = \frac{4zn_2 P}{\pi \omega_x^3 \omega_y} \quad \frac{1}{f_y} = \frac{4zn_2 P}{\pi \omega_y^3 \omega_x} \tag{A.5}
\]

For \( \omega_x = \omega_y = \omega \), Eq.(A.5) reduces back into Eq.(5.2)
Appendix B

Gaussian beam propagation equations in Kerr medium

In the following, we present the differential equations for a non-circular Gaussian beam with $\omega_x \neq \omega_y$, propagating through a Kerr medium.

The propagation of a Gaussian beam through a Kerr medium is represented by Eq.8.34. This equation does not include the coupling in Kerr lensing between the sagittal and tangential planes. Therefore, Eq.8.34 is solved for each plane separately, i.e. the resonator is split into two uncoupled resonators: one with $f/\cos \theta$ and $L/n$, and the other with $f \cdot \cos \theta$ and $L/n^3$. As shown in Ch.8.3.1, the uncoupled case is further simplified by using the transformed complex beam parameter $q^{-1}$, hence one does not need to solve Eq.8.34 step by step inside the Kerr medium. This calculation provides qualitatively accurate results which are sufficient for the design and realization of KLM TiS cavities.

\footnote{See Ap.A}
For a quantitatively accurate result, one must take into account the coupling effect in Kerr lensing between the sagittal and tangential planes. Thus, Eq. 8.33 can be rewritten as:

$$\frac{d}{dz} \left( \frac{1}{q_x} \right) + \left( \frac{1}{q_x} \right)^2 + \frac{d}{dz} \left( \frac{1}{f_x} \right) = 0 \quad (B.1)$$

Substituting Eq. A.5 into Eq. B.1 yields:

$$\frac{d}{dz} \left( \frac{1}{q_x} \right) + \left( \frac{1}{q_x} \right)^2 + \left( \frac{\lambda}{\pi} \right)^2 \frac{K}{\omega_x^3 \omega_y} = 0, \quad (B.2)$$

and similar equation can be obtained for the $y$ plane by replacing $x \leftrightarrow y$.

Here, there is no transformation for the complex beam parameter so that Eq. B.2 will be reduced into a free-space propagation equation. Therefore, one must solve Eq. B.2 step by step inside the crystal. Substituting Eq. 8.13 into Eq. B.2 and separating into real and imaginary parts, provides two coupled equations for the mode spot size $\omega_x$ and the radius of curvature $R_x$:

$$\frac{d}{dz} \left( \frac{1}{R_x} \right) = \left( \frac{\lambda}{\pi} \right)^2 \frac{\omega_x^3}{\omega_y} \left( 1 - K \frac{\omega_x}{\omega_y} \right) - \left( \frac{1}{R_x} \right)^2 \quad (B.3)$$

$$\frac{d}{dz} \omega_x = \omega_x \left( \frac{1}{R_x} \right)$$

These two equations are coupled with one another and with the two equations for the perpendicular plane. We can solve these equations numerically.
Appendix C

Measuring the parameters of a prism pair

In order to calculate $P(\lambda, Z, h)$ in Eq.9.3 for a given prism pair on the optical table with an apex angle $\alpha$ and refractive index $n_p(\lambda)$, one must measure the values of $Z$ and $h$. Obviously, it is inaccurate to measure these values directly on the optical table, as can be seen from Fig.9.2. However, measuring the separation between the prisms tips $L_p$ is by far more accurate. Therefore, one can set the beam to pass exactly at the tip of both prisms and measure the wavelength $\lambda_T$. As illustrated in Fig.C.1, the values of $Z$ and $h$ can be calculated immediately:

$$Z = L_p \sin \theta_{out}$$

$$h = L_p \sin \theta_{out}$$

(C.1)

The angle $\theta_{out}$ can be geometrically calculated:

$$\theta_{out} = \left( \frac{\alpha}{2} + \frac{\pi}{2} - \arcsin[n_p(\lambda_T) \sin \theta^*] \right)$$

$$\theta^* = \alpha - \arcsin\left( \frac{\sin \theta_B}{n_p(\lambda_T)} \right)$$

(C.2)

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Chapter C. Measuring the parameters of a prism pair

Figure C.1: The values of $Z$ and $h$ for a given prism pair can be calculated by measuring the separation length $L_p$ between the prisms tips and setting the beam with a known wavelength $\lambda_T$ to pass at the tip of both prisms.

Once $Z$ and $h$ are calculated, one can calculate how much $h$ needs to be changed (by translating the prisms) in order to compensate for dispersion. This method (Eq. C.1 and Eq. C.2) is sufficient for cavity design in order to achieve mode locking. However, this method assumes that the beam is geometrical, i.e. a straight line with no width. In reality, a beam with radius $\omega$ will propagate through a finite, non-zero length at the tip of the prism. If the beam size is known, one can calculate the length that the beam center passes within the prism tips, given by:

$$d_i = \frac{\sin \alpha}{\sin \theta^* \cos \theta_B} \omega_i$$

where $i = 1, 2$ stands for each prism. Hence, the beam passes a total length of $d_1 + d_2$ in both tips. To calculate $Z$ and $h$ one needs to solve:

$$L_p^2 = Z^2 + h^2$$

$$d(\lambda_T, Z, h) = d_1 + d_2$$
where \( d(\lambda, Z, h) \) is the length inside both prisms, which can be geometrically calculated.
Bibliography


תקציר

מגבר אופטי של וניסיוניים מחקרים תיאורטיים החיבור מתאר אופנים-נעול (לייזר) לייזר מבוסס גביש: פרטב. מבוסס גביש \( \text{Kerr} \) אפקט קר "אופנים ע-ספיר נעול-טיטניום". המ Reaper האפקט קר התוכתי של אופטוס ואופטוס קצרים, הנושב בוליש חשבוני כדי לחקור תהליכים מחלימים פיסי-קור, כימיה ובוליזוניקה.

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- השלב השני: הפסקoles של הלייזרים, באירות הגישה של המ러ות המחוז
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בשלב השליםו, אנו מגדירים מגזים בוליצים של הדגמאות מחולק לא-ליגנאר
מנזר, ובר的一种 מגזים בוליצים של בוליצים יוצרים מזקלת
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